LIMIT MODELS IN STRICTLY STABLE ABSTRACT ELEMENTARY CLASSES

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Abstract. In this paper, we examine the locality condition for non-splitting and determine the level of uniqueness of limit models that can be recovered in some stable, but not superstable, abstract elementary classes. In particular we prove:

THEOREM 1.2. *Suppose that $\mathcal{K}$ is an abstract elementary class satisfying*

1. *the joint embedding and amalgamation properties with no maximal model of cardinality μ.*
2. *stability in μ.*
3. *$\kappa^*_\mu(\mathcal{K}) < \mu^+$.*
4. *continuity for non-μ-splitting (i.e. if $p \in \text{ga-S}(M)$ and M is a limit model witnessed by $\langle M_i \mid i < \alpha \rangle$ for some limit ordinal $\alpha < \mu^+$ and there exists $N \prec M_0$ so that $p \restriction M_i$ does not μ-split over N for all $i < \alpha$, then p does not μ-split over N).*

For θ and δ limit ordinals $< \mu^+$ both with cofinality $\geq \kappa^*_\mu(\mathcal{K})$, if $\mathcal{K}$ satisfies symmetry for non-μ-splitting (or just (μ, δ)-symmetry), then, for any M_1 and M_2 that are (μ, θ) and (μ, δ)-limit models over M_0, respectively, we have that M_1 and M_2 are isomorphic over M_0.

Note that no tameness is assumed.

§1. Introduction. Because the main test question for developing a classification theory for abstract elementary classes (AECs) is Shelah's Categoricity Conjecture [1, Problem D.1], the development of independence notions for AECs has often started with an assumption of categoricity ([11, 21, 20] and others). Consequently, the independence relations that result are superstable or stronger (see, for instance, good λ-frames and the superstable prototype [13, Example II.3.(A)]). However, little progress has been made to understand stable, but not superstable AECs. A notable exception is the work on κ-coheir of Boney and Grossberg [3], which only requires stability in the guise of 'no weak κ-order property.' In this paper, we add to the understanding of strictly stable AECs with a different approach and under different assumptions than [3]. In particular, our analysis uses towers and the standard definition of Galois-stability. Moreover, we work without assuming any of the strong locality assumptions (tameness, type shortness, etc.) of [3]. We hope that this work will lead to further exploration in this context.

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The main tool in our analysis is a tower, which was first conceived to study superstable AECs (see, for instance [14] or [15]). The 'right analogue' of superstability in AECs has been the subject of much research. Shelah has commented that this notion suffers from 'schizophrenia,' where several equivalent concepts in first-order seem to bifurcate into distinct notions in nonelementary settings; see the recent Grossberg and Vasey [7] for a discussion of the different possibilities (and a suprising proof that they are equivalent under tameness).

Common to much analysis of superstable AECs is the uniqueness of limit models. Uniqueness of limit models was first proved to follow from a categoricity assumption in [10, 12, 14, 15, 16]. Later, μ-superstability, which was isolated by Grossberg, VanDieren, and Villaveces [6, Assumption 2.8(4)], was shown to imply uniqueness of limit models under the additional assumption of μ-symmetry [17]. μ-superstability was modeled on the local character characterization of superstability in first-order and was already known to follow from categoricity [14]. The connection between μ-symmetry and structural properties of towers [17] inspired recent research on μ-superstable classes: [18, 19]. Moreover, years of work culminating in the series of papers [14, 15, 16, 17, 18, 19] has led to the extraction of a general scheme for proving the uniqueness of limit models (note that amalgamation is generally assumed in these papers, but this is not true of [14, 15, 16]). In this paper we witness the power of this new scheme by adapting the technology developed in [17] to cover μ-stable, but not μ-superstable classes. We suspect that this new technology of towers will likely be used to answer other problems in classification theory (in both first order and non-elementary settings).

This paper focuses on the question to what degree the uniqueness of limit models can be recovered if we assume the class is Galois-stable in μ, but not μ-superstable, by refocusing the question from "*Are all* (μ,α)-limit models isomorphic (over the base)?" to "*For which* $\alpha,\beta<\mu^+$ *are* (μ,α)-limit models and (μ,β)-limit models isomorphic (over the base)?" Based on first-order results (summarized in [6, Section 2]), we have the following conjecture.

Conjecture 1.1. *Suppose $\mathcal{K}$ is an AEC with μ-amalgamation and is μ-stable. The set*

$$\{\alpha<\mu^+:\mathrm{cf}(\alpha)=\alpha \textit{ and } (\mu,\alpha)\textit{-limit models are isomorphic to } (\mu,\mu)\textit{-limit models}\}$$

is a non-trivial interval of regular cardinals. Moreover, the minimum of this set, denoted by $\kappa^*_\mu(\mathcal{K})$, is an important measure of the complexity of $\mathcal{K}$.

Our main result (restated from the abstract) proves this conjecture under certain assumptions.

Theorem 1.2. *Suppose that $\mathcal{K}$ is an abstract elementary class satisfying*

1. *the joint embedding and amalgamation properties with no maximal model of cardinality μ.*
2. *stabilty in μ.*
3. *$\kappa^*_\mu(\mathcal{K})<\mu^+$.*
4. *continuity for non-μ-splitting (i.e. if $p\in\text{ga-S}(M)$ and M is a limit model witnessed by $\langle M_i\mid i<\alpha\rangle$ for some limit ordinal $\alpha<\mu^+$ and there exists N*

so that $p \restriction M_i$ does not μ-split over N for all $i < \alpha$, then p does not μ-split over N).

For θ and δ limit ordinals $< \mu^+$ both with cofinality $\geq \kappa^*_\mu(\mathcal{K})$, if $\mathcal{K}$ satisfies symmetry for non-μ-splitting (or just (μ, δ)-symmetry), then, for any M_1 and M_2 that are (μ, θ) and (μ, δ)-limit models over M_0, respectively, we have that M_1 and M_2 are isomorphic over M_0.

Assumption 2.3 collects these assumptions together, and we discuss them following that statement. In this statement, the "measure of complexity" from Conjecture 1.1 is $\kappa^*_\mu(\mathcal{K})$, a generalization of the first-order $\kappa(T)$ (see Definition 2.1). An important feature of this work is that it explores the underdeveloped field of strictly stable AECs.

We end with a short comment contextualizing this paper within the body of work on limit models. The general arguments for investigating the uniqueness of limit models have appeared before (see [15, 6]). One use is that they give a version of saturated models without dealing with smaller models and give a sense of how difficult it is to create saturated models. Many works of AECs take a 'local approach' of analyzing $\mathcal{K}_\lambda$ (the models of size λ) to derive structure on $\mathcal{K}_{\lambda^+}$ (see [13, Chapter II] or [11] for the most prominent examples). Because not even the existence of models of size $< \lambda$ is assumed, Galois saturation (which quantifies over smaller models) cannot be used, and limit models have become the standard substitute. Moreover, we expect that limit models will take on a greater importance in the context of strictly stable AECs, especially those without assumption of tameness. Of the various analogues for AECs (see [7, Theorem 1.2]), most have seen extensive analysis, but only in the context of tameness. One of the remaining notions (solvability; see [13, Chapter IV]) seems to have no weakening to the strictly stable context. What remains are μ-superstability and the uniqueness of limit models. Thus, it is reasonable to assume that understanding strictly stable AECs will require understanding the connection between 'μ-stability' (Assumption 2.3 here) and limit models. Theorem 1.2 is a step towards this understanding.

After circulating this paper but before publication, Vasey and Mazari-Armida used our results to make further progress in the field. Vasey used Theorem 1.2 in his work to characterize stable AECs [22], especially in terms of unions of sufficiently saturated models being saturated [22, Theorem 11.11]. Additionally, Vasey [22, Theorem 3.7] gives some natural conditions for Assumption 2.3.(4) below, which he calls the weak continuity of splitting. On the other hand, Mazari-Armida identified naturally occuring strictly stable AECs. By analyzing limit models of different cofinalities, he demonstrated that the class of torsion-free abelian groups and the class of finitely Butler groups, both with the pure subgroup relation, are strictly stable AECs [9].

Section 2 reviews key definitions and facts with Assumption 2.3 being the key hypotheses throughout the paper. Section 3 discusses the notion of relatively full towers. Section 4 discusses reduced towers and proves the key lemma, Theorem 4.5. Section 5 concludes with a proof of the main theorem, Theorem 1.2.

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§2. Background. We refer the reader to [1], [5], [6], [15], and [17] for definitions and notations of concepts such as Galois-stability, μ-splitting, etc. We reproduce a few of the more specialized definitions and results here.

Grossberg, VanDieren, and Villaveces [6, Assumption 2.8] isolated a notion they call 'μ-superstability'[1] by examining consequences of categoricity from [10] and [14]. The key feature in this assumption is that there are no infinite splitting chains (as forbidden in [14, Theorem 2.2.1]). We weaken μ-superstability by only forbidding long enough splitting chains. How long is 'long enough' is measured by $\kappa^*_\mu(\mathcal{K})$, which is a relative of [5, Definition 4.3] and universal local character [3, Definition 3.5]. Following [3], we add the '*' to this symbol to denote that the chain is required to have the property that M_{i+1} is universal over M_i.

Definition 2.1. *We define $\kappa^*_\mu(\mathcal{K})$ to be the minimal, regular $\kappa < \mu^+$ so that for every increasing and continuous sequence $\langle M_i \in \mathcal{K}_\mu \mid i \leq \alpha\rangle$ with $\alpha \geq \kappa$ regular which satisfies for every $i < \alpha$, M_{i+1} is universal over M_i, and for every non-algebraic $p \in \text{ga-S}(M_\alpha)$, there exists $i < \alpha$ such that p does not μ-split over M_i. If no such κ exists, we say $\kappa^*_\mu(\mathcal{K}) = \infty$.*

We call $\kappa^*_\mu(\mathcal{K})$ the 'universal local character for μ-nonsplitting for $\mathcal{K}$,' or simply the 'universal local character' for short when μ and $\mathcal{K}$ are fixed.

In [5, Theorem 4.13], Grossberg and VanDieren show that if $\mathcal{K}$ is a tame stable abstract elementary class satisfying the joint embedding and amalgamation properties with no maximal models, then there exists a single bound for $\kappa^*_\mu(\mathcal{K})$ for all sufficiently large μ in which $\mathcal{K}$ is μ-stable. This proof works by considering the χ-order property of Shelah. We can also give a direct bound assuming tameness.

Proposition 2.2. *Let $\mathcal{K}$ be an AEC with amalgamation that is λ-stable and (λ, μ)-tame. Then $\kappa^*_\mu(\mathcal{K}) \leq \lambda$.*

Note that the proof does not require the extensions to be universal.

Proof. Let $\langle M_i \in K_\mu : i \leq \alpha\rangle$ be an increasing, continuous chain with $\text{cf}(\alpha) \geq \lambda$ and $p \in \text{ga-S}(M_\alpha)$. By [10, Claim 3.3.(1)] and λ-stability, there is $N_0 \prec M_\alpha$ of size λ such that p does not λ-split over N_0. By tameness, p does not μ-split over N_0. By the cofinality assumption, there is $i_* < \alpha$ such that $N_0 \prec M_{i_*}$. By monotonicity, p does not μ-split over M_{i_*}. $\dashv$

This definition motivates our main assumption. We use this collection only to group these items together and will explicitly list Assumption 2.3 when it is part of a result's hypothesis.

Assumption 2.3.

1. *$\mathcal{K}$ satisfies the joint embedding and amalgamation properties with no maximal model of cardinality μ.*
2. *$\mathcal{K}$ is stable in μ.*
3. *$\kappa^*_\mu(\mathcal{K}) < \mu^+$.*

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[1]We do not use this here, but the definition of μ-superstability strengthens Assumption 2.3 by requiring that $\kappa^*_\mu(\mathcal{K})$ be ω.

4. $\mathcal{K}$ satisfies (limit) continuity for non-μ-splitting (i.e. if $p \in$ ga-S(M) and M is a limit model witnessed by $\langle M_i \mid i < \theta\rangle$ for some limit ordinal $\theta < \mu^+$ and there exists N so that $p \restriction M_i$ does not μ-split over N for all $i < \theta$, then p does not μ-split over N).

A few comments on the assumption is in order. Note that tameness is not assumed in this paper. Amalgamation is commonly assumed in the study of limit models, although [14, 15, 16] replace it with more nuanced results about amalgamation bases. Stability in μ is necessary for the conclusion of Theorem 1.2 to make sense; otherwise, there are no limit models! We have argued (both in principle and in practice) that varying the local character cardinal is the right generalization of superstability to stability in this context. However, we have kept the "continuity cardinal" to be ω; this is the content of Assumption 2.3.(4). This seems necessary for the arguments[2]. It seems reasonable to hope that some failure of continuity for non-splitting will lead to a nonstructure result, but this has not yet been achieved.

The assumptions are (trivially) satisfied in any superstable AEC and, therefore, any categorical AEC. However, in this context, the result is already known. For a new example, we look to the context of strictly stable homogeneous structures as developed in Hyttinen [8, Section 1]. In the homogeneous contexts, Galois types are determined by syntactic types. Armed with this, Hyttinen studies the normal syntactic notion of nonsplitting under a stable, unsuperstable hypothesis [8, Assumption 1.1], and shows that syntactic splitting satisfies continuity and (more than) the universal local character of syntactic nonsplitting is $\aleph_1$.[3] It is easy to see that the syntactic version of nonsplitting implies our nonsplitting, which already implies $\kappa^*_\mu(\mathcal{K}) = \aleph_1$. The following argument shows that, if N is limit over M, the converse holds as well, which is enough to get the limit continuity for our semantic definition of splitting. Since the context of homogeneous model theory is very tame, we don't worry about attaching a cardinal to non-splitting because they are all equivalent.

Suppose that N is a limit model over M, witnessed by $\langle M_i \mid i < \alpha\rangle$, and $p \in$ ga-S(M) syntactically splits over M. Then, since Galois types are syntactic, there are $b, c \in N$ such that ga-tp$(b/M) =$ ga-tp(c/M) and, for an appropriate ϕ, $\phi(x, b, m) \wedge \neg\phi(x, c, m) \in p$. We can find $\beta, \beta' < \alpha$ such that $b \in N_\beta$ and $c \in N_{\beta'}$. Since b and c have the same type, we can find an amalgam $N_* \succ N_\beta$ and $f : N_\alpha \to_M N_*$ such that $f(b) = c$. Since N is universal over $N_{\beta'}$, we can find $h : N_* \to_{N'_\beta} N$. This gives us an isomorphism $h \circ f : N_\beta \cong h(f(N_\beta))$ and we claim that this witnesses the semantic version splitting: $c \in N_{\beta'}$, so $c = h(c) = h(f(b)) \in h(f(N_\beta))$ and, thus, $\neg\phi(x, c, m) \in p \restriction h(f(N_\beta))$. On the other hand, $\phi(x, c, m) = h \circ f(\phi(x, b, m)) \in h \circ f(p \restriction N_\beta)$. Thus, we have witnessed $h \circ f(p \restriction N_\beta) \neq p \restriction h(f(N_\beta))$.

Note if $\kappa^*_\mu(\mathcal{K}) = \mu$, then the conclusion of Theorem 1.2 is uninteresting, but the results still hold: any two limit models whose lengths have the same cofinality

[2] The first author claimed in the discussion following [2, Lemma 9.1] that only long continuity was necessary. However, after discussion with Sebastien Vasey, this seems to be an error.

[3] It shows that it is at most $\aleph_1$. However, if it were $\aleph_0$, the class would be superstable, contradicting the assumption.

are isomorphic on general grounds. Also, we assume joint embedding, etc. only in $\mathcal{K}_\mu$. However, to simplify presentation, we work as though these properties held in all of $\mathcal{K}$ and, thus, we work inside a monster model. This will allow us to write ga-tp(a/M) rather than ga-tp$(a/M;N)$ and witness Galois type equality with automorphisms. The standard technique of working inside of a (μ,μ^+)-limit model can translate our proofs to ones not using a monster model.

Under these assumptions, it is possible to construct towers. This is the key technical tool in this construction. Towers were introduced in Shelah and Villaveces [14] and expanded upon in [15] and subsequent works.

Recall that, if I is well-ordered, then it has a successor function which we will denote $+1$ (or $+_I 1$ if necessary). Also, we typically restrict our attention to well-ordered I.

Definition 2.4 ([15, Definition I.5.1]).

1. *A* tower indexed by I in $\mathcal{K}_\mu$ *is a triple* $\mathcal{T} = \langle \bar{M}, \bar{a}, \bar{N}\rangle$ *where*
   - $\bar{M} = \langle M_i \in \mathcal{K}_\mu \mid i \in I\rangle$ *is an increasing sequence of limit models;*
   - $\bar{a} = \langle a_i \in M_{i+1}\backslash M_i \mid i+1 \in I\rangle$ *is a sequence of elements;*
   - $\bar{N} = \langle N_i \in K_\mu \mid i+1 \in I\rangle$ *such that* $N_i \prec M_i$ *with* M_i *universal over* N_i*; and*
   - ga-tp(a_i/M_i) *does not* μ*-split over* N_i*.*
2. *A tower* $\mathcal{T} = \langle \bar{M}, \bar{a}, \bar{N}\rangle$ *is* continuous *iff* $\bar{M}$ *is, i. e.,* $M_i = \cup_{j<i} M_j$ *for all limit* $i \in I$*.*
3. $\mathcal{K}^*_{\mu,I}$ *is the collection of all towers indexed by* I *in* $\mathcal{K}_\mu$*.*

Note that continuity is not required of all towers.

We will switch back and forth between the notation $\mathcal{K}^*_{\mu,\alpha}$ where α is an ordinal and $\mathcal{K}^*_{\mu,I}$ where I is a well ordered set (of order type α) when it will make the notation clearer. When we deal with relatively full towers, we will find the notation using I to be more convenient for book-keeping purposes.

For $\beta < \alpha$ and $\mathcal{T} = (\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ we write $\mathcal{T} \restriction \beta$ for the tower made up of the sequences $\bar{M} \restriction \beta := \langle M_i \mid i < \beta\rangle$, $\bar{a} \restriction \beta := \langle a_i \mid i+1 < \beta\rangle$, and $\bar{N} \restriction \beta := \langle N_i \mid i+1 < \beta\rangle$.

We will construct increasing chains of towers. Here we define what it means for one tower to extend another:

Definition 2.5. *For* I *a sub-ordering of* I' *and towers* $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I}$ *and* $(\bar{M}', \bar{a}', \bar{N}') \in \mathcal{K}^*_{\mu,I'}$*, we say*

$$(\bar{M}, \bar{a}, \bar{N}) \leq (\bar{M}', \bar{a}', \bar{N}')$$

if $\bar{a} = \bar{a}' \restriction I$*,* $\bar{N} = \bar{N}' \restriction I$*, and for* $i \in I$*,* $M_i \preceq_{\mathcal{K}} M'_i$ *and whenever* M'_i *is a proper extension of* M_i*, then* M'_i *is universal over* M_i*. If for each* $i \in I$*,* M'_i *is universal over* M_i *we will write* $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}', \bar{a}', \bar{N}')$*.*

For γ a limit ordinal $< \mu^+$ and $\langle I_j \mid j < \gamma\rangle$ a sequence of well ordered sets with I_j a sub-ordering of I_{j+1}, if $\langle(\bar{M}^j, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,I_j} \mid j < \gamma\rangle$ is a $<$-increasing sequence of towers, then the union $\mathcal{T}$ of these towers is determined by the following:

- for each $\beta \in \bigcup_{j<\gamma} I_j$, $M_\beta := \bigcup_{\beta \in I_j;\, j<\gamma} M^j_\beta$

- the sequence $\langle a_\beta \mid \exists(j<\gamma)\ \beta+1,\beta\in I_j\rangle$, and
- the sequence $\langle N_\beta \mid \exists(j<\gamma)\ \beta+1,\beta\in I_j\rangle$

is a tower in $\mathcal{K}^*_{\mu,\bigcup_{j<\gamma} I_j}$, provided that $\mathcal{K}$ satisfies the continuity property for non-μ-splitting and that $\bigcup_{j<\gamma} I_j$ is well ordered. Note that it is our desire to take increasing unions of towers that leads to the necessity of the continuity property.

We also need to recall a few facts about directed systems of partial extensions of towers that are implicit in [15]. These are helpful tools in the inductive construction of towers and are used in other work (see, e.g., [17, Facts 2 and 3]): Fact 2.6 will get us through the successor step of inductive constructions of directed systems, and Fact 2.7 describes how to pass through the limit stages. An explicit proof of Fact 2.7 appears as [17, Fact 3], and we provide a proof of Fact 2.6 below. Two important notes:

- These facts do not require that the towers be continuous.
- The work in [15] does not assume amalgamation, so more care had to be taken in working with large limit models (in place of the monster model) and towers made of amalgamation bases. The amalgamation assumption in this (and other) papers significantly simplifies the situation.

FACT 2.6 ([15]). *Suppose $\mathcal{T}$ is a tower in $\mathcal{K}^*_{\mu,\alpha}$ and $\mathcal{T}'$ is a tower of length $\beta<\alpha$ with $\mathcal{T}\restriction\beta<\mathcal{T}'$, if $f\in\mathrm{Aut}_{M_\beta}(\mathfrak{C})$ and M''_β is a limit model universal over M_β such that* ga-tp(a_β/M''_β) *does not μ-split over N_β and $f(\bigcup_{i<\beta} M'_i)\prec_{\mathcal{K}} M''_\beta$, then the tower $\mathcal{T}''\in\mathcal{K}^*_{\mu,\beta+1}$ defined by $f(\mathcal{T}')$ concatenated with the model M''_β, element a_β and submodel N_β is an extension of $\mathcal{T}\restriction(\beta+1)$.*

PROOF. This is a routine verification from the definitions. $\mathcal{T}''\restriction\beta$ is isomorphic to the tower $\mathcal{T}'$ and we are given the required nonsplitting and that, for $i<\beta$, $f(M'_i)\prec M''_\beta$, so we have that $\mathcal{T}''\in\mathcal{K}^*_{\mu,\beta+1}$. Similarly, $f\ \mathcal{T}\restriction\beta$, so $\mathcal{T}\restriction\beta<\mathcal{T}'$ implies $\mathcal{T}\restriction\beta<\mathcal{T}''\restriction\beta$. To extend this to $\mathcal{T}\restriction(\beta+1)<\mathcal{T}''\restriction(\beta+1)=\mathcal{T}''$, we note that M''_β is universal over M_β by assumption. ⊣

FACT 2.7 ([15]). *Fix $\mathcal{T}\in\mathcal{K}^*_{\mu,\alpha}$ for α a limit ordinal. Suppose $\langle\mathcal{T}^i\in\mathcal{K}^*_{\mu,i}\mid i<\alpha\rangle$ and $\langle f_{i,j}\mid i\le j<\alpha\rangle$ form a directed system of towers. Suppose*

- *each $\mathcal{T}^i$ extends $\mathcal{T}\restriction i$*
- *$f_{i,j}\restriction M_i=id_{M_i}$*
- *M^{i+1}_{i+1} is universal over $f_{i,i+1}(M^i_i)$.*

Then there exists a direct limit $\mathcal{T}^\alpha$ and mappings $\langle f_{i,\alpha}\mid i<\alpha\rangle$ to this system so that $\mathcal{T}^\alpha\in\mathcal{K}^*_{\mu,\alpha}$, $\mathcal{T}^\alpha$ extends $\mathcal{T}$, and $f_{i,\alpha}\restriction M_i=id_{M_i}$.

Finally, to prove results about the uniqueness of limit models, we will additionally need to assume that non-μ-splitting satisfies a symmetry property over limit models. We refine the definition of symmetry from [17, Definition 3] for non-μ-splitting; this localization only requires symmetry to hold when M_0 is (μ,δ)-limit over N.

DEFINITION 2.8. *Fix $\mu\ge\mathrm{LS}(\mathcal{K})$ and δ a limit ordinal $<\mu^+$. We say that an abstract elementary class exhibits* (μ,δ)-symmetry for non-μ-splitting *if whenever models $M,M_0,N\in\mathcal{K}_\mu$ and elements a and b satisfy the conditions 1-4 below,* *then there exists M^b a limit model over M_0, containing b, so that* ga-tp(a/M^b) *does not μ-split over N. See Figure 1.*

1. *M is universal over M_0 and M_0 is a (μ, δ)-limit model over N.*
2. *$a \in M \backslash M_0$.*
3. ga-tp(a/M_0) *is non-algebraic and does not μ-split over N.*
4. ga-tp(b/M) *is non-algebraic and does not μ-split over M_0.*

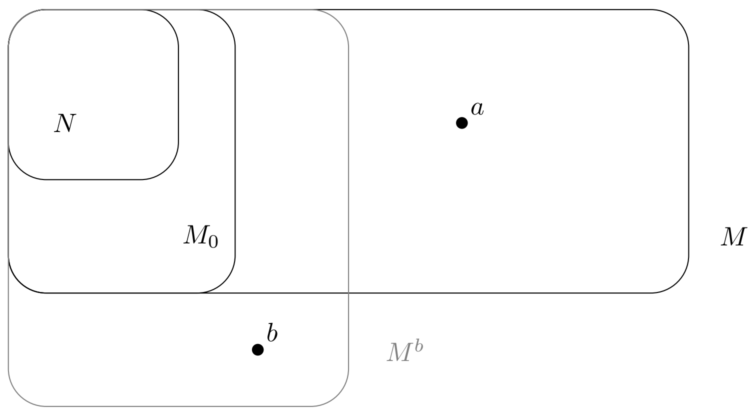

FIGURE 1. A diagram of the models and elements in the definition of (μ, δ)-symmetry. We assume the type ga-tp(b/M) does not μ-split over M_0 and ga-tp(a/M_0) does not μ-split over N. Symmetry implies the existence of M^b a limit model over M_0 so that ga-tp(a/M^b) does not μ-split over N.

Note that (μ, δ)-symmetry is the same as $(\mu, \mathrm{cf}\, \delta)$-symmetry.

§3. Relatively Full Towers. One approach to proving the uniqueness of limit models is to construct a continuous relatively full tower of length θ, and then conclude that the union of the models in this tower is a (μ, θ)-limit model. In this section we confirm that this approach can be carried out in this context, even if we remove continuity along the relatively full tower.

DEFINITION 3.1 ([14, Definition 3.2.1]). *For M a (μ, θ)-limit model, let*

$$\mathfrak{St}(M) := \left\{ (p, N) \;\middle|\; \begin{array}{l} N \prec_{\mathcal{K}} M; \\ N \textit{ is a } (\mu,\theta)\textit{-limit model}; \\ M \textit{ is universal over } N; \\ p \in \text{ga-S}(M) \textit{ is non-algebraic} \\ \textit{and } p \textit{ does not } \mu\textit{-split over } N. \end{array} \right\}$$

Elements of $\mathfrak{St}(M)$ *are called* strong types. *Two strong types* $(p_1, N_1) \in \mathfrak{St}(M_1)$ *and* $(p_2, N_2) \in \mathfrak{St}(M_2)$ *are* parallel *iff for every M' of cardinality μ extending M_1 and M_2 there exists $q \in$* ga-S(M') *such that q extends both p_1 and p_2 and q does not μ-split over N_1 nor over N_2.*

Definition 3.2 (Relatively Full Towers). *Suppose that I is a well-ordered set. Let $(\bar{M},\bar{a},\bar{N})$ be a tower indexed by I such that each M_i is a (μ,σ)-limit model. For each i, let $\langle M_i^\gamma \mid \gamma<\sigma\rangle$ witness that M_i is a (μ,σ)-limit model.*
The tower $(\bar{M},\bar{a},\bar{N})$ is full relative to $(M_i^\gamma)_{\gamma<\sigma,i\in I}$ *iff*

1. *there exists a cofinal sequence $\langle i_\alpha \mid \alpha<\theta\rangle$ of I of order type θ such that there are $\mu\cdot\omega$ many elements between i_α and $i_{\alpha+1}$ and*
2. *for every $\gamma<\sigma$ and every $(p,M_i^\gamma)\in\mathfrak{St}(M_i)$ with $i_\alpha\leq i<i_{\alpha+1}$, there exists $j\in I$ with $i\leq j<i_{\alpha+1}$ such that $(\text{ga-tp}(a_j/M_j),N_j)$ and (p,M_i^γ) are parallel.*

The following proposition will allow us to use relatively full towers to produce limit models. The fact that relatively full towers yield limit models was first proved in [15] and in [6] and later improved in [4, Proposition 4.1.5]. We notice here that the proof of [4, Proposition 4.1.5] does not require that the tower be continuous and does not require that $\kappa^*_\mu(\mathcal{K})=\omega$. We provide the proof for completeness.

Proposition 3.3 (Relatively full towers provide limit models). *Let θ be a limit ordinal $<\mu^+$ satisfying $\theta=\mu\cdot\theta$. Suppose that I is a well-ordered set as in Definition 3.2.(1).*

Let $(\bar{M},\bar{a},\bar{N})\in\mathcal{K}^*_{\mu,I}$ be a tower made up of (μ,σ)-limit models, for some fixed σ with $\kappa^*_\mu(\mathcal{K})\leq\text{cf}(\sigma)<\mu^+$. If $(\bar{M},\bar{a},\bar{N})\in\mathcal{K}^*_{\mu,I}$ is full relative to $(M_i^\gamma)_{i\in I,\gamma<\sigma}$, then $M:=\bigcup_{i\in I}M_i$ is a (μ,θ)-limit model over M_{i_0}.

Proof. Because the sequence $\langle i_\alpha \mid \alpha<\theta\rangle$ is cofinal in I and $\theta=\mu\cdot\theta$, we can rewrite $M:=\bigcup_{i\in I}M_i=\bigcup_{\beta<\theta}M_{i_\beta}=\bigcup_{\alpha<\theta}\bigcup_{\delta<\mu}M_{i_{\mu\alpha+\delta}}$.

For $\alpha<\theta$ and $\delta<\mu$, notice

$$M_{i_{\mu\alpha+\delta+1}} \text{ realizes every type over } M_{i_{\mu\alpha+\delta}}. \tag{1}$$

To see this take $p\in\text{ga-S}(M_{i_{\mu\alpha+\delta}})$. By our assumption that $\text{cf}(\sigma)\geq\kappa^*_\mu(\mathcal{K})$, p does not μ-split over $M^\gamma_{i_{\mu\alpha+\delta}}$ for some $\gamma<\sigma$. Therefore $(p,M^\gamma_{i_{\mu\alpha+\delta}})\in\mathfrak{St}(M_{i_{\mu\alpha+\delta}})$. By definition of relatively full towers, there is an a_k with $i_{\mu\alpha+\delta}\leq k<i_{\mu\alpha+\delta+1}$ so that $(\text{ga-tp}(a_k/M_k),N_k)$ and $(p,M^\gamma_{i_{\mu\alpha+\delta}})$ are parallel. Because $M_{i_{\mu\alpha+\delta}}\prec_{\mathcal{K}}M_k$, by the definition of parallel strong types, it must be the case that $a_k\models p$.

By a back and forth argument we can conclude from (1) that $M_{i_{\mu\alpha+\mu}}$ is universal over $M_{i_{\mu\alpha}}$. Thus M is a (μ,θ)-limit model.

To see the details of the back-and-forth argument mentioned in the previous paragraph, first translate (1) to the terminology of [1]: (1) witnesses that $\bigcup_{\beta<\mu}M_{i_{\mu\alpha+\beta}}$ is 1-special over $M_{i_{\mu\alpha}}$. Then, refer to the proof of Lemma 10.5 of [1].

$\dashv$

§4. Reduced Towers. The proof of the uniqueness of limit models from [10, 6, 15, 16] is two dimensional. In the context of towers, the relatively full towers are used to produce a (μ,θ)-limit model, but to conclude that this model is also a (μ,ω)-limit model, a $<$-increasing chain of ω-many continuous towers of length $\theta+1$ is constructed. We adapt this construction to prove Theorem 1.2. Instead of creating a chain of ω-many towers, we produce a chain of δ-many towers, and instead of each tower in this chain being continuous, we only require that these towers are continuous at limit ordinals of cofinality at least $\kappa^*_\mu(\mathcal{K})$.

The use of towers should be compared with the proof uniqueness of limit models in [13, Section II.4] (details are given in [2, Section 9]). Both proofs create a 'square' of models, but do so in a different way. The proof here will proceed by starting with a 1-dimensional tower of models and then, in the induction step, extend this tower to fill out the square. In contrast, the induction step of [13, Lemma II.4.8] adds single models at a time. This seems like a minor distinction (or even just a difference in how the induction step is carried out), but there is a real distinction in the resulting squares. In [13], the construction is 'symmetric' in the sense that θ and δ are treated the same. However, in the proof presented here, this symmetry is broken and one could 'detect' which side of the square was laid out initially by observing where continuity fails.

In [6, 15, 16, 17], the continuity of the towers is achieved by restricting the construction to reduced towers, which under the stronger assumptions of [6, 15, 16, 17] are shown to be continuous. We take this approach and notice that continuity of reduced towers at certain limit ordinals can be obtained with the weaker assumptions of Theorem 1.2, in particular $\kappa^*_\mu(\mathcal{K}) < \mu^+$.

Definition 4.1. *A tower* $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ *is said to be* reduced *provided that for every* $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ *with* $(\bar{M}, \bar{a}, \bar{N}) \leq (\bar{M}', \bar{a}, \bar{N})$ *we have that for every* $i < \alpha$,

$$(*)_i \quad M'_i \cap \bigcup_{j<\alpha} M_j = M_i.$$

The proofs of the following three results about reduced towers only require that the class $\mathcal{K}$ be stable in μ and that μ-splitting satisfies the continuity property. Although [14] works under stronger assumptions than we currently, none of these results use anything beyond Assumption 2.3. In particular, $\kappa^*_\mu(\mathcal{K}) = \omega$ holds in [14], but is not used.

Fact 4.2 ([14, Theorem 3.1.13]). *Let* $\mathcal{K}$ *satisfy Assumption 2.3. There exists a reduced* $<$*-extension of every tower in* $\mathcal{K}^*_{\mu,\alpha}$.

Fact 4.3 ([14, Theorem 3.1.14]). *Let* $\mathcal{K}$ *satisfy Assumption 2.3. Suppose* $\langle (\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}^*_{\mu,\alpha} \mid \gamma < \beta \rangle$ *is a* $<$*-increasing and continuous sequence of reduced towers such that the sequence is continuous in the sense that for a limit* $\gamma < \beta$, *the tower* $(\bar{M}, \bar{a}, \bar{N})^\gamma$ *is the union of the towers* $(\bar{M}, \bar{a}, \bar{N})^\zeta$ *for* $\zeta < \gamma$. *Then the union of the sequence of towers* $\langle (\bar{M}, \bar{a}, \bar{N})^\gamma \in \mathcal{K}^*_{\mu,\alpha} \mid \gamma < \beta \rangle$ *is itself a reduced tower.*

In fact the proof of Fact 4.3 gives a slightly stronger result which allows us to take the union of an increasing chain of reduced towers of increasing index sets and conclude that the union is still reduced.

Fact 4.4 ([6, Lemma 5.7]). *Let* $\mathcal{K}$ *satisfy Assumption 2.3. Suppose that* $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ *is reduced. If* $\beta < \alpha$, *then* $(\bar{M}, \bar{a}, \bar{N}) \restriction \beta$ *is reduced.*

The following theorem is related to [17, Theorem 3], which additionally assumes that $\kappa^*_\mu(\mathcal{K}) = \omega$; in other words it assumes $\mathcal{K}$ μ-superstable. Instead, we allow for strict stability (that is, $\kappa^*_\mu(\mathcal{K})$ to be uncountable) at the cost of only guaranteeing continuity at limits of large cofinality. In particular, the proof is similar to the proof of $(a) \to (b)$ in [17, Theorem 3], but we crucially allow our towers to be discontinuous at γ where $\mathrm{cf}(\gamma) < \kappa^*_\mu(\mathcal{K})$. We provide the details where the proof differs.

THEOREM 4.5. *Suppose $\mathcal{K}$ satisfies Assumption 2.3. Let α be an ordinal and δ be a limit ordinal so that $\kappa^*_\mu(\mathcal{K}) \leq \mathrm{cf}(\delta) < \alpha$. If $\mathcal{K}$ satisfies (μ, δ)-symmetry for non-μ-splitting and $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ is reduced, then the tower $(\bar{M}, \bar{a}, \bar{N})$ is continuous at δ (i.e., $M_\delta = \bigcup_{\beta<\delta} M_\beta$).*

PROOF. Suppose the theorem is false. Then we can find a reduced tower $\mathcal{T} := (\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ that is a counterexample of minimal length at δ in the sense that:

1. $M_\delta \neq \cup_{i<\delta} M_i$ and
2. if $(\bar{M}', \bar{a}', \bar{N}') \in \mathcal{K}^*_{\mu,\alpha'}$ is reduced and discontinuous at δ, then $\alpha \leq \alpha'$.

Notice that Fact 4.4 implies that $\alpha = \delta + 1$. Let $b \in M_\delta \backslash \bigcup_{i<\delta} M_i$ witness the discontinuity of the tower at δ.

By Fact 4.2 and Fact 4.3, we can build $\mathcal{T}^i = (\bar{M}^i, \bar{a}^i, \bar{N}^i) \in \mathcal{K}^*_{\mu,\delta}$ for $i \leq \delta$ such that $\mathcal{T}^0 = \mathcal{T} \restriction \delta$ and $\langle \mathcal{T}^i \mid i \leq \delta \rangle$ is a $<$-increasing, continuous chain. By δ-applications of Fact 4.2 in between successor stages of the construction, we can require that for $\beta < \delta$

$$\begin{gathered} M^{i+1}_\beta \text{ is a } (\mu, \delta)\text{-limit over } M^i_\beta \\ \text{and consequently } M^{i+1}_\beta \text{ is a } (\mu, \delta)\text{-limit over } N_\beta. \end{gathered} \tag{2}$$

Let $M^\delta_{diag} := \bigcup_{i<\delta,\ \beta<\delta} M^i_\beta$. Figure 2 is an illustration of these models.

There are two cases depending on whether b is in M^δ_{diag} or not. Both cases lead to a contradiction of our assumption that $\mathcal{T}$ is reduced.

Case 1: $b \in M^\delta_{diag}$

The first case will contradict our assumption that $(\bar{M}, \bar{a}, \bar{N})$ is reduced. We have that $\mathcal{T}^\delta$ is an extension of $\mathcal{T} \restriction \delta$ and that M^δ_{diag} contains b. Let M^δ_δ be an extension of M^δ_{diag} that is also a universal extension of M_δ. Then $\mathcal{T}^\delta{}^\frown\langle M^\delta_\delta \rangle$ is an extension of $\mathcal{T}$. Since $b \in M^\delta_{diag}$, there is some $j < \delta$ so $b \in M^\delta_j$. Because $\mathcal{T}$ is reduced, we have that

$$M^\delta_j \cap \bigcup_{i<\alpha} M_i = M_j.$$

Notice that the $M^\delta_j \cap M_\delta$ on the LHS contains b, but the RHS does not contain b, a contradiction.

Case 2: $b \notin M^\delta_{diag}$

Then $\text{ga-tp}(b/M^\delta_{diag})$ is non-algebraic. Consider the sequence $\langle \check{M}_i \mid i < \delta \rangle$ defined by $\check{M}_i := M^i_i$ if i is a successor and $\check{M}_i := \bigcup_{j<i} M^j_j$ for i a limit ordinal. Notice that (2) implies that this sequence witnesses that M^δ_{diag} is a (μ, δ)-limit model. Because M^δ_{diag} is a (μ, δ)-limit model, by our assumption that $\mathrm{cf}(\delta) \geq \kappa^*_\mu(\mathcal{K})$ and monotonicity of non-splitting, there exists a successor ordinal $i^* < \delta$ so that

$$\text{ga-tp}(b/M^\delta_{diag}) \text{ does not } \mu\text{-split over } M^{i^*}_{i^*}. \tag{3}$$

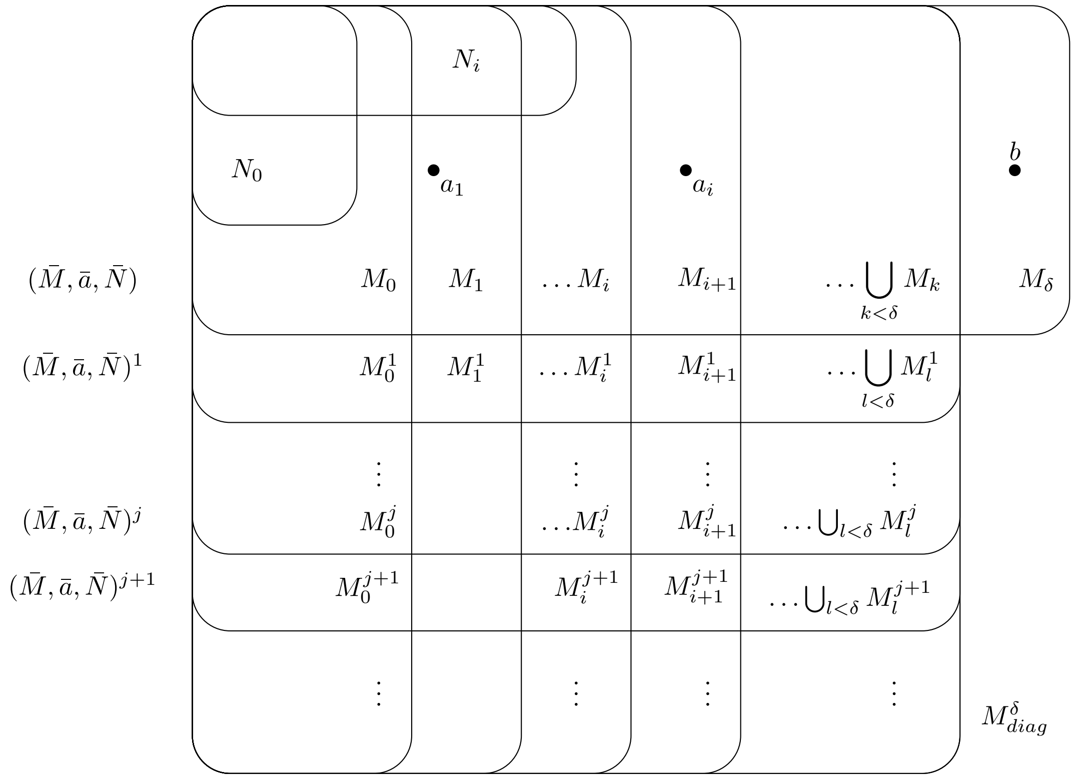

FIGURE 2. $(\bar{M}, \bar{a}, \bar{N})$ and the towers $(\bar{M}, \bar{a}, \bar{N})^j$ extending $(\bar{M}, \bar{a}, \bar{N}) \restriction \delta$.

Our next step in Case (2) is to consider the tower formed by the diagonal elements in Figure 2. In particular, let $\mathcal{T}^{diag}$ be the sequence $(M_i^i, a_i, N_i)_{i<\delta}$. We claim that $\mathcal{T}^{diag} \in \mathcal{K}^*_{\mu,\delta}$ and that $\mathcal{T}^{diag}$ extends $\mathcal{T} \restriction \delta$.

We will now use $\mathcal{T}^{diag}$ to construct a tower containing b that extends $\mathcal{T} \restriction \delta$. First we find an approximation, $\mathcal{T}^b$, which is a tower of length i^*+1 that contains b and extends $\mathcal{T}^{diag} \restriction (i^*+2)$. Then through a directed system of mappings, we move this tower so that the result is as desired.

To define $\mathcal{T}^b$, first notice that by (2), $M_{i^*}^{i^*}$ is a (μ,δ)-limit over N_{i^*}. Now, referring to the Figure 1, apply (μ,δ)-symmetry to a_{i^*} standing in for a, $M_{i^*}^{i^*}$ representing M_0, N_{i^*} as N, M_{diag}^δ as M, and b as itself. We can conclude that there exists M^b containing b, a limit model over $M_{i^*}^{i^*}$, for which ga-tp(a_{i^*}/M^b) does not μ-split over N_{i^*}. Define the tower $\mathcal{T}^b \in \mathcal{K}^*_{\mu,i^*+2}$ by the sequences $\bar{a} \restriction (i^*+1)$, $\bar{N} \restriction (i^*+1)$ and $\bar{M}'$ with $M'_j := M_j^j$ for $j \leq i^*$ and $M'_{i^*+1} := M^b$. Notice that $\mathcal{T}^b$ is an extension of $\mathcal{T}^{diag} \restriction (i^*+2)$ containing b.

Next, we will explain how we can use this tower to find a tower $\mathring{\mathcal{T}}^\delta \in \mathcal{K}^*_{\mu,\delta}$ extending $\mathcal{T}^{diag}$ with $b \in \bigcup_{j<\delta} \mathring{M}_j^\delta$. This will be enough to contradict our assumption that $\mathcal{T}$ was reduced.

We want to build $\langle \mathring{\mathcal{T}}^j, f_{j,k} \mid i^*+2 \leq j \leq k \leq \delta\rangle$ a directed system of towers so that for $j \geq i^*+2$

1. $\mathring{\mathcal{T}}^{i^*+2} = \mathcal{T}^b$
2. $\mathring{\mathcal{T}}^j \in \mathcal{K}^*_{\mu,j}$ for $j \leq \delta$
3. $\mathcal{T}^{diag} \restriction j \leq \mathring{\mathcal{T}}^j$ for $j \leq \delta$
4. $f_{j,k}(\mathring{\mathcal{T}}^j) \leq \mathring{\mathcal{T}}^k \restriction j$ for $j \leq k < \delta$
5. $f_{j,k} \restriction M^j_j = id_{M^j_j}$ $j \leq k < \delta$
6. $\mathring{M}^{j+1}_{j+1}$ is universal over $f_{j,j+1}(\mathring{M}^j_j)$ for $j < \delta$
7. $b \in \mathring{M}^j_{i^*+1}$ for $j \leq \delta$
8. ga-tp$(f_{j,k}(b)/M^k_k)$ does not μ-split over $M^{i^*}_{i^*}$ for $j < k < \delta$.

Construction: We will define this directed system by induction on k, with $i^*+2 \leq k \leq \alpha$. The base and successor case are exactly as in the proof of Theorem 5 of [17]. The only difference in the construction here is at limit stages in which $\mathcal{T}^{diag}$ is not continuous. Therefore we will concentrate on the details of the construction for stage k and $k+1$ where $k < \delta$ is a limit ordinal for which $\mathcal{T}^{diag}$ is discontinuous at k.

Construction, Case 1: k is limit where $\mathcal{T}^{diag}$ is discontinuous.
First, let $\grave{\mathcal{T}}^k$ and $\langle \grave{f}_{j,k} \mid i^*+2 \leq j < k\rangle$ be a direct limit of the system defined so far. We use the $\grave{}$ notation since these are only approximations to the tower and mappings that we are looking for. We will have to take some care to find a direct limit that contains b in order to satisfy Condition 7 of the construction. By Fact 2.7 and our induction hypothesis, we may choose this direct limit so that for all $j < k$

$$\grave{f}_{j,k} \restriction M^j_j = id_{M^j_j}.$$

Consequently $\grave{M}^\alpha_j := \grave{f}_{j,k}(\mathring{M}^j_j)$ is universal over M^j_j, and $\bigcup_{j<k} \grave{M}^k_j$ is a limit model witnessed by Condition 6 of the construction. Additionally, the tower $\grave{\mathcal{T}}^k$ composed of the models $\grave{M}^k_j$, extends $\mathcal{T}^{diag} \restriction k$.

We will next show that for every $j < k$,

$$\text{ga-tp}(\grave{f}_{i^*+2,k}(b)/M^j_j) \text{ does not } \mu\text{-split over } M^{i^*}_{i^*}. \tag{4}$$

To see this, recall that for every $j < k$, by the definition of a direct limit, $\grave{f}_{i^*+2,k}(b) = \grave{f}_{j,k}(f_{i^*+2,j}(b))$. By Condition 8 of the construction, we know

$$\text{ga-tp}(f_{i^*+2,j}(b)/M^j_j) \text{ does not } \mu\text{-split over } M^{i^*}_{i^*}.$$

Applying $\grave{f}_{j,k}$ to this implies ga-tp$(\grave{f}_{i^*+2,k}(b)/M^j_j)$ does not μ-split over $M^{i^*}_{i^*}$, establishing (4).

Because M^{j+1}_{j+1} is universal over M^j_j by construction, we can apply the continuity of non-splitting to (4), yielding

$$\text{ga-tp}(\grave{f}_{i^*+2,k}(b)/\bigcup_{j<k} M^j_j) \text{ does not } \mu\text{-split over } M^{i^*}_{i^*}. \tag{5}$$

Because $\grave{f}_{i^*+2,k}$ fixes $M^{i^*+1}_{i^*+1}$, ga-tp$(b/M^{i^*+1}_{i^*+1}) =$ ga-tp$(\grave{f}_{i^*+2,k}(b)/M^{i^*+1}_{i^*+1})$. We can then apply the uniqueness of non-splitting extensions (see [15, Theorem I.4.12]) to (5) to see that ga-tp$(\grave{f}_{i^*+2,k}(b)/\bigcup_{j<k} M^j_j) =$ ga-tp$(b/\bigcup_{j<k} M^j_j)$. Thus

we can fix g an automorphism of the monster model fixing $\bigcup_{j<k} M_j^j$ so that $g(\grave{f}_{i^*+2,k}(b)) = b$.

We will then define $\mathring{\mathcal{T}}^k$ to be the tower $g(\grave{\mathcal{T}}^k)$, and the mappings for our directed system will be $f_{j,k} := g \circ \grave{f}_{j,k}$ for all $i^* + 2 \leq j < k$.

Notice that by our induction hypothesis we have that $b \in \mathring{M}^{i^*+2}_{i^*+1}$. Then, by definition of a direct limit we have $\grave{f}_{i^*+2,k}(b) \in \grave{M}^k_{i^*+1}$. Therefore $g(\grave{f}_{i^*+2,k}(b)) = b \in \mathring{M}^k_{i^*+1}$, satisfying Condition 7 of the construction. Furthermore for all $j < k$, we have that $f_{j,k}(b) = b$. Therefore by (3) and monotonicity of non-splitting, Condition 8 of the construction holds.

Notice that $\mathcal{T}^{diag}$ being discontinuous at k does not impact this stage of the construction since we only require that $\mathring{\mathcal{T}}^k$ be a tower of length k and therefore $\mathring{\mathcal{T}}^k$ need not contain models extending M^k_k. The discontinuity plays a role at the next stage of the construction.

Construction, Case 2: $k + 1$ is successor of limit where $\mathcal{T}^{diag}$ is discontinuous.
Suppose that $\mathcal{T}^{diag}$ is discontinuous at k and that $\mathring{\mathcal{T}}^k \in \mathcal{K}^*_{\mu,k}$ has been defined.

By our choice of i^*, we have ga-tp$(b/\bigcup_{l<\alpha} M^l_l)$ does not μ-split over $M^{i^*}_{i^*}$. So in particular by monotonicity of non-splitting, we notice:

$$\text{ga-tp}(b/M^{k+1}_k) \text{ does not } \mu\text{-split over } M^{i^*}_{i^*}. \tag{6}$$

Using the definition of towers (i.e. M^{k+1}_k is a (μ, δ)-limit over N_k and ga-tp(a_k/M^{k+1}_k) does not μ-split over N_k) and the choice of i^*, we can apply (μ, δ)-symmetry to a_k, M^{k+1}_k, $\bigcup_{l<\delta} M^l_l$, b and N_k which will yield M^b_k a limit model over M^{k+1}_k containing b so that ga-tp(a_k/M^b_k) does not μ-split over N_k (see Figure 3).

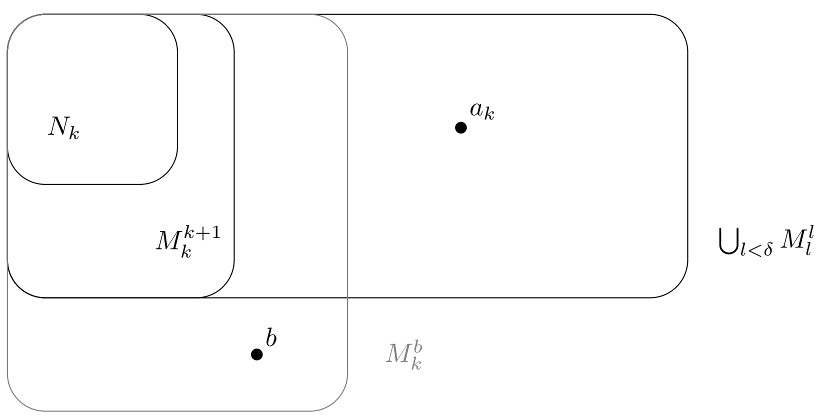

FIGURE 3. A diagram of the application of (μ, δ)-symmetry in the successor stage of the directed system construction in the proof of Theorem 4.5. We have ga-tp$(b/\bigcup_{l<\delta} M^l_l)$ does not μ-split over M^{k+1}_k and ga-tp(a_k/M^{k+1}_k) does not μ-split over N_k. Symmetry implies the existence of M^b_k a limit model over M^{k+1}_k. so that ga-tp(a_k/M^b) does not μ-split over N_k.

Notice that M^b_k has no relationship to $\mathring{\mathcal{T}}^k$. In particular, it does not contain $\bigcup_{l<k}\mathring{M}^l_l$. Fix M' to be a model of cardinality μ extending both $\bigcup_{l<k}\mathring{M}^l_l$ and M^{k+1}_k. Since M^b_k is a limit model over M^{k+1}_k which is a limit model over M^k_k, there exits $f : M' \to M^{k+1}_k$ with $f = id_{M^k_k}$ so that M^b_k is also universal over $f(\bigcup_{l<k}\mathring{M}^l_l)$. Because ga-tp$(b/M^k_k)$ does not μ-split over $M^{i^*}_{i^*}$ and f fixes M^k_k, we know that ga-tp$(f(b)/M^k_k)$ does not μ-split over $M^{i^*}_{i^*}$. But because $f(b)$ and b both realize the same types over $M^{i^*+1}_{i^*+1}$, we can conclude by the uniqueness of non-splitting extensions that ga-tp$(f(b)/M^k_k) =$ ga-tp(b/M^k_k); so there is $g \in \mathrm{Aut}_{M^k_k}(\mathfrak{C})$ with $g(f(b)) = b$. Since M^b_k is universal over M^k_k and $b \in M^b_k$, we can choose g so that $g(f(M')) \prec_{\mathcal{K}} M^b_k$.

Take $\mathring{M}^{k+1}_k$ to be an extension of M^b_k which is also universal over M^{k+1}_{k+1}, and set $f_{k,k+1} := g \circ f$. To see that Condition 8 of the construction holds, just apply monotonicity and the fact that $f_{k,k+1}(b) = b$ to (3). See figure 4.

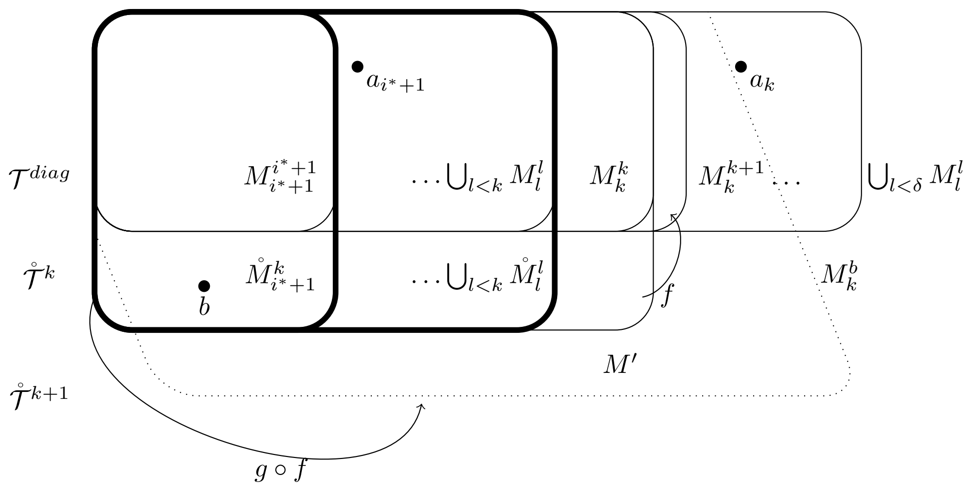

FIGURE 4. The construction of $\mathring{\mathcal{T}}^{k+1}$(dotted) from $\mathring{\mathcal{T}}^k$ (bold) with $g \circ f$ fixing M^k_k and b.

It is easy to check by invariance and the induction hypothesis that $\mathring{\mathcal{T}}^{k+1}$ defined by the models $\mathring{M}^{k+1}_l := f_{k,k+1}(\mathring{M}^k_l)$ for $l < k$ satisfies the remaining requirements on $\mathring{\mathcal{T}}^{k+1}$. Then the rest of the directed system can be defined by the induction hypothesis and the mappings $f_{l,k+1} := f_{l,k} \circ f_{k,k+1}$ for $i^* + 2 \leq l < k$.

This completes the construction.

Case (2), continued: Now that we have a tower $\mathring{\mathcal{T}}^\delta$ extending $\mathcal{T} \restriction \delta$ which contains b, we are in a situation similar to the proof in Case (1). To contradict that $\mathcal{T}$ is reduced, we need only lengthen $\mathring{\mathcal{T}}^\delta$ to a discontinuous extension of the entire tower $(\bar{M}, \bar{a}, \bar{N})$ by taking the δ^{th} model to be some extension of $\bigcup_{i<\delta}\mathring{M}^i_i$

which is also universal over M_δ. This discontinuous extension of $(\bar{M}, \bar{a}, \bar{N})$ along with $b \in \mathring{M}^{\delta}_{i^*+1}$ witness that $(\bar{M}, \bar{a}, \bar{N})$ cannot be reduced.

$\dashv$

Although not used here, the converse of this theorem is also true, as in [17]. Note that the following does not have any assumption about $\kappa^*_\mu(\mathcal{K})$.

Proposition 4.6. *Suppose $\mathcal{K}$ satisfies Assumption 2.3.(1), (2), and (4). Suppose further that that, for every reduced tower $(\bar{M}, \bar{a}, \bar{M}) \in \mathcal{K}^*_{\mu,\alpha}$, $\bar{M}$ is continuous at limit ordinals of cofinality δ. Then $\mathcal{K}$ satisfies (μ, δ)-symmetry for non μ-splitting.*

Proof. The proof is an easy adaptation of [17, Theorem 3.$(b) \to (a)$]. The same argument works; the only adaptations are to require that every limit model to in fact be a (μ, δ) limit model and that the tower $\mathcal{T}$ be of length $\delta + 1$[4]. $\dashv$

§5. Uniqueness of Long Limit Models. We now begin the proof Theorem 1.2, which we restate here.

Theorem 1.2. *Suppose that $\mathcal{K}$ is an abstract elementary class satisfying Assumption 2.3. For θ and δ limit ordinals $< \mu^+$ both with cofinality $\geq \kappa^*_\mu(\mathcal{K})$, if $\mathcal{K}$ satisfies symmetry for non-μ-splitting (or just (μ, δ)-symmetry), then, for any M_1 and M_2 that are (μ, θ) and (μ, δ)-limit models over M_0, respectively, we have that M_1 and M_2 are isomorphic over M_0.*

The structure of the proof of Theorem 1.2 from this point on is similar to the proof in [6, Theorem 1.9]. For completeness we include the details here, and emphasize the points of departure from [6, Theorem 1.9].

We construct an array of models which will produce a model that is both a (μ, θ)- and a (μ, δ)-limit model. Let θ be an ordinal as in the definition of relatively full tower so that $\mathrm{cf}(\theta) \geq \kappa^*_\mu(\mathcal{K})$ and let $\delta = \kappa^*_\mu(\mathcal{K})$. The goal is to build an array of models with $\delta + 1$ rows so that the bottom row of the array is a relatively full tower indexed by a set of cofinality $\theta + 1$ continuous at θ. To do this, we will be adding elements to the index set of towers row by row so that at stage n of our construction the tower that we build is indexed by I_n described here.

The index sets I_β will be defined inductively so that $\langle I_\beta \mid \beta < \delta + 1 \rangle$ is an increasing and continuous chain of well-ordered sets. We fix I_0 to be an index set of order type $\theta + 1$ and will denote it by $\langle i_\alpha \mid \alpha \leq \theta \rangle$. We will refer to the members of I_0 by name in many stages of the construction. These indices serve as anchors for the members of the remaining index sets in the array. Next we demand that for each $\beta < \delta$, $\{j \in I_\beta \mid i_\alpha < j < i_{\alpha+1}\}$ has order type $\mu \cdot \beta$ such that each I_β has supremum i_θ. An example of such $\langle I_\beta \mid \beta \leq \delta \rangle$ is $I_\beta = \theta \times (\mu \cdot \beta) \bigcup \{i_\theta\}$ ordered lexicographically, where i_θ is an element $\geq$ each $i \in \bigcup_{\beta<\delta} I_\beta$. Also, let $I = \bigcup_{\beta<\delta} I_\beta$.

To prove Theorem 1.2, we need to prove that, for a fixed $M \in \mathcal{K}$ of cardinality μ, any (μ, θ)-limit and (μ, δ)-limit model over M are isomorphic over M. Since all (μ, θ)-limits over M are isomorphic over M (and the same holds for (μ, δ)-limits),

[4] In a happy coincidence, the notation in that proof already agrees with this change.

it is enough to construct a single model that is simultaneously (μ,θ)-limit and (μ,δ)-limit over M. Let us begin by fixing a limit model $M \in \mathcal{K}_\mu$. We define, by induction on $\beta \leq \delta$, a $<$-increasing and continuous sequence of towers $(\bar{M},\bar{a},\bar{N})^\beta$ such that

1. $\mathcal{T}^0 := (\bar{M},\bar{a},\bar{N})^0$ is a tower with $M^0_0 = M$.
2. $\mathcal{T}^\beta := (\bar{M},\bar{a},\bar{N})^\beta \in \mathcal{K}^*_{\mu,I_\beta}$.
3. For every $(p,N) \in \mathfrak{St}(M^\beta_i)$ with $i_\alpha \leq i < i_{\alpha+1}$ there is $j \in I_{\beta+1}$ with $i_\alpha < j < i_{\alpha+1}$ so that $(\text{ga-tp}(a_j/M^{\beta+1}_j), N^{\beta+1}_j)$ and (p,N) are parallel.

See Figure 5.

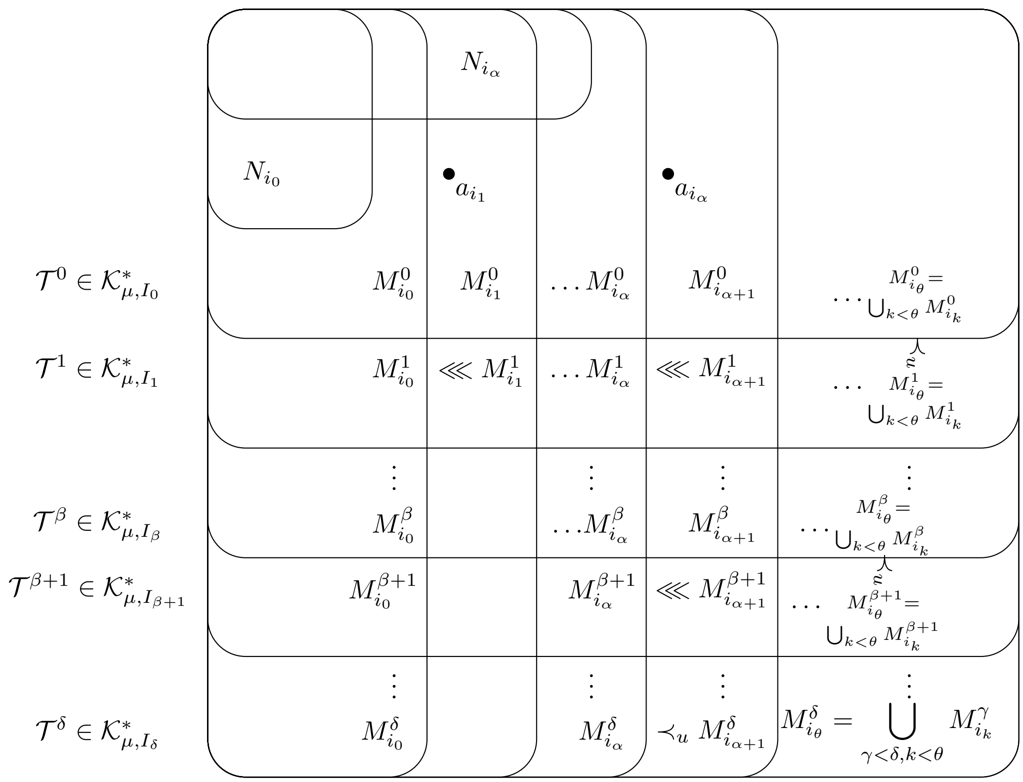

FIGURE 5. The chain of length δ of towers of increasing index sets I_j of cofinality $\theta+1$. The symbol $\lll$ indicates that there are μ many new indices between i_β and $i_{\beta+1}$ in $I_{j+1}\backslash I_j$. The elements indexed by these indices realize all the strong types over the model $M^j_{i_\alpha}$. The notation $\prec_u$ is an abbreviation for a universal extension.

Given M, we can find a tower $(\bar{M},\bar{a},\bar{N})^0 \in \mathcal{K}^*_{\mu,I_0}$ with $M \preceq_{\mathcal{K}} M^0_0$ because of the existence of universal extensions and because $\kappa^*_\mu(\mathcal{K}) < \mu^+$. At successor stages we first take an extension of $(\bar{M},\bar{a},\bar{N})^\beta$ indexed by $I_{\beta+1}$ and realizing all the strong types over the models in $(\bar{M},\bar{a},\bar{N})^\beta$. This tower may not be reduced, but by Fact 4.2, it has a reduced extension. At limit stages take unions of the chain of towers defined so far.

Notice that by Fact 4.3, the tower $\mathcal{T}^\delta$ formed by the union of all the $(\bar{M}, \bar{a}, \bar{N})^\beta$ is reduced. Furthermore, by Theorem 4.5 every one of the reduced towers $\mathcal{T}^j$ is continuous at θ because $\mathrm{cf}(\theta) \geq \kappa^*_\mu(\mathcal{K})$. Therefore $M^\delta_{i_\theta} = \bigcup_{k<\theta} M^\delta_{i_k}$, and by the definition of the ordering $<$ on towers, the last model in this tower $(M^\delta_{i_\theta})$ is a (μ, δ)-limit model witnessed by $\langle M^j_{i_\theta} \mid j < \delta\rangle$. Since $M^1_{i_\theta}$ is universal over M, we have that $M^\delta_{i_\theta}$ is (μ, δ)-limit over M.

Next to see that $M^\delta_{i_\theta}$ is also a (μ, θ)-limit model, notice that $\mathcal{T}^\delta$ is relatively full by condition 3 of the construction and the same argument as [6, Claim 5.11]. Therefore by Theorem 4.5 and our choice of δ with $\mathrm{cf}(\delta) \geq \kappa^*_\mu(\mathcal{K})$, the last model $M^\delta_{i_\theta}$ in this relatively full tower is a (μ, θ)-limit model over M.

This completes the proof of Theorem 1.2.

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