COHEIR IN AVERAGEABLE CLASSES

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This document serves as a supplement to Boney [Bon, Section 3], in particular to provide the details to Theorems 3.9 and 3.10. Familiarity with that paper and Boney and Grossberg [BG] is assumed and the reader interested in motivation should consult one of those papers.

Definition 1. *K is averageable iff there is a collection of first order formulas $\mathcal{F} := \mathcal{F}_K$ and Γ such that*

- *for all $\phi \in p \in \Gamma$, $\neg\phi \in \mathcal{F}$;*
- *$M \prec_K N$ iff $M \subset N$ and for all $\mathbf{m} \in M$ and $\phi(\mathbf{x}) \in \mathcal{F}$,*
$$M \vDash \phi[\mathbf{m}] \iff N \vDash \phi[\mathbf{m}]$$
- *given $\{M_i \in K : i \in I\}$ and an ultrafilter U on I, $\Pi^\Gamma M_i/U \in K$ and this Γ-ultraproduct satisfies Łoś' Theorem for the formulas in $\mathcal{F}$; and*
- *each $M \in K$ omits each $p \in \Gamma$.*

There are two types of coheir to consider. The first is Galois coheir ${}^g\!\downarrow$ (or maybe s-coheir[1]). In this case, we consider Galois types over finite domains. When Galois types are syntactic, these are complete syntactic types over a finite set. The second is t-coheir ${}^t\!\downarrow$, which is more like the first order version. This formulation was not available in [BG] since there was no logic to work with. For averageable classes, $\mathcal{F}_K$ gives us an appropriate notion of formula to work with.

Definition 2. (1) *Given $A, B, C \subset M$, we say $A \overset{g}{\underset{C}{\overset{M}{\downarrow}}} B$ iff*

for all finite $a \in A, b \in B, c \in C$, $gtp(a/bc)$ is realized in C.

(2) *K has the weak Galois order property iff there are finite tuples $\langle a_i, b_i \in M : i < \omega\rangle$ and c and types $p \neq q \in gS(c)$ such that, for all $i, j < \omega$,*
$$j < i \implies a_i b_j \vDash q$$
$$j \geq i \implies a_i b_j \vDash p$$
(3) *Given $A, B, C \subset M$, we say $A^t \underset{C}{\overset{M}{\downarrow}} B$ iff*

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[1]The s and t come from Shelah [Sh:c]; s means the *s*et of parameters is finite and the t means the *t*ype is finite.

for all finite $a \in A, b \in B, c \in C$ and $\phi(x, y, z) \in \mathcal{F}_K$, if $M \vDash \phi(a, b, c)$, then there is $c' \in C$ such that $M \vDash \phi(c', b, c)$.

(4) *K has the weak order property iff there are finite tuples $\langle a_i, b_i \in M : i < \omega\rangle$ and a formula $\phi(x, y, c) \in \mathcal{F}_K$ with $c \in M$ such that, for all $i, j < \omega$,*

$$j < i \iff M \vDash \phi(a_i, b_j, c)$$

Suppose that averageable K has amalgamation and does not have the weak order property. Note that we have begun talking about Galois types over sets (rather than models, as standard) even though we only have amalgamation over models. This adds some additional dificulties, but we are careful to avoid them here. The adjective 'weak' in describing the order property means that we only require ω length orders, rather than all ordinal lengths as in Shelah [Sh394].

We first work through the properties of ${}^g\!\!\downarrow$.

Proposition 3 ([BG].4.3g)**.** *If C is given such that*

$$gtp(a/C; M) = gtp(a'/C; M') \text{ and } a \,{}^g\!\!\underset{C}{\overset{M}{\downarrow}}\, b \text{ and } b \,{}^g\!\!\underset{C}{\overset{M'}{\downarrow}}\, a'$$

then $gtp(ab/C; M) = gtp(a'b/C; M')$.

Note that a, a', b might be finite tuples, not just singletons. Indeed, by the finite character of types and nonforking, this immediately follows for arbitrary sets.

Proof: Deny. By $< \omega$-type shortness, there is finite $c \in C$ such that $gtp(ab/c; M) = gtp(a'b/c; M')$. Set $p = gtp(ab/c; M)$ and $q = gtp(a'b/c; M')$. We build $\langle a_i, b_i \in C : i < \omega\rangle$ such that

(1) $a_i b \vDash p$;
(2) $\forall j < i, a_i b_j \vDash q$;
(3) $a b_i \vDash q$; and
(4) $\forall j \geq i, a_i b_j \vDash p$.

Note that (2) and (4) give the weak order property.

Construction: Suppose we have constructed $\{a_j, b_j : j < i\}$ so far. Since $a \,{}^g\!\!\underset{C}{\overset{M}{\downarrow}}\, b$, $gtp(a/cb\{b_j : j < i\}; M)$ is realized by some $a_i \in C$. To witness this, there are $N \succ M$ and $f : M \to_{cb\{b_j:j<i\}} N$ such that $f(a) = a_i \in C$. This gives

$$\begin{aligned} gtp(ab/c; M) &= gtp(a_i b/c; M) \\ gtp(ab_j/c; M) &= gtp(a_i b_j/c; M) \forall j < i \end{aligned}$$

Similarly, $b \,{}^g\!\!\underset{C}{\overset{M'}{\downarrow}}\, a'$, so $gtp(b/ca'\{a_j : j \leq i\}; M')$ is realized by some $b_i \in C$. This gives

$$\begin{aligned} gtp(a'b_i/c; M') &= gtp(a'b/c; M') \\ gtp(a_j b_i/c; M') &= gtp(a_j b/c; M') \forall j < i \end{aligned}$$

†

Proposition 4 ([BG].8.2g). *$^g\!\downarrow$ satisfies Extension over models.*

Proof: Suppose $A^g \underset{M}{\overset{N}{\downarrow}} M$ and $M \subset B \subset N$. Because of this nonforking, there is an index set I and an ultrafiler U on I such that, if $h : M \to \Pi^\Gamma M/U$ is the ultrapower map, then $h(gtp(A/M;N))$ is realized in $\Pi^\Gamma M/U$. Call this realization A'. Let h^+ be an isomorphism with range $\Pi^\Gamma N/U$ that contains h and set $M^{\Gamma,U} = (h^+)^{-1}(\Pi^\Gamma M/U)$ and $N^{\Gamma,U}$ similarly. Now we claim $M^{\Gamma,U\,g} \underset{M}{\overset{N^{\Gamma,U}}{\downarrow}} B$. Let $a \in M^{\Gamma,U}, b \in B, c \in M$. Set $p = gtp(a/bc;N)$ and $[f]_U = (h^+)^{-1}(a)$. Then

$$\begin{array}{rcl} a & \vDash & p \\ f_U & \vDash & h(p) \\ \{i \in I & : & f(i) \vDash p\} \in U \end{array}$$

Take i_0 from this U-large set. Since $[f]_U \in \Pi^\Gamma M/U$, we have $f(i) \in M$ and $f(i) \vDash p$. Thus, by Monotonicity, $(h^+)^{-1}(A') \underset{M}{\overset{g}{\downarrow}} b$. †

Proposition 5 ([BG].4.1g). (1) *If $^g\!\downarrow$ has Existence and Extension, then it has Symmetry.*
(2) *If $^g\!\downarrow$ has Symmetry, then it has Uniqueness.*

Proof: We want to show that $A^g \underset{M}{\overset{N}{\downarrow}} B$ iff $B^g \underset{M}{\overset{N}{\downarrow}} A$; it is enough to show this for finite a and b. Suppose $a^g \underset{M}{\overset{N}{\downarrow}} b$. Then, by Existence and Extension, there is some $b' \in N'$ such that $gtp(b'/M;N') = gtp(b/M;N)$ and $b'^g \underset{M}{\overset{N'}{\downarrow}} a$. Thus, we have $gtp(ab'/M;N') = gtp(ab/M;N)$ by Proposition 3. By Invariance, $b^g \underset{M}{\overset{N}{\downarrow}} a$.

Now we want to show Uniqueness. Let $gtp(A/M;N) = gtp(A'/M;N')$, $A^g \underset{M}{\overset{N}{\downarrow}} B$, and $A'^g \underset{M}{\overset{N'}{\downarrow}} B$. By Symmetry, we have $B^g \underset{M}{\overset{N'}{\downarrow}} A'$. By Proposition 3, we have $gtp(AB/M;N) = gtp(A'B/M;N')$.

Theorem 6 ([BG].5.1g). *If K is an averageable class with amalgamation that doesn't have the weak Galois order property and every model is $\aleph_0$-Galois saturated*[2]*, then $^g\!\downarrow$ is an independence relation in the sense of [BG].*

[2]This is equivalent to Existence holding

Proof: Existence is provided by the hyptothesis and the rest of the properties follow by the above. Note that the base of ${}^g\!\perp$ is necessarily a model in Existence, Extension, Symmetry, and Uniqueness. †

Now we work through the properties of ${}^t\!\perp$.

Note that the following requires that $\mathcal{F}$ is closed under conjunctions and negations. The first is easy to acheive (the $\mathcal{F}$ associated to an averageable class can be so closed with no loss), while the second is much harder. However, it holds of our two main cases: Γ-closed and Γ-nice data.

Proposition 7 ([BG].4.3t)**.** *Suppose that $\mathcal{F}$ is closed under conjunctions and negations. If C is given such that*

$$tp_{\mathcal{F}}(a/C;M) = tp_{\mathcal{F}}(a'/C;M') \text{ and } a\,{}^t\!\underset{C}{\overset{M}{\perp}} b \text{ and } b\,{}^t\!\underset{C}{\overset{M'}{\perp}} a'$$

then $tp_{\mathcal{F}}(ab/C;M) = tp_{\mathcal{F}}(a'b/C;M')$.

Note that a, a', b might be finite tuples, not just singletons. Indeed, by the finite character of types and nonforking, this immediately follows for arbitrary sets.

Proof: Deny. There is $c \in C$ and $\phi(x,y,c)$ such that $M \vDash \phi(a,b,c)$ and $M' \vDash \neg\phi(a',b,c)$. We build $\langle a_i, b_i \in C : i < \omega \rangle$ such that

(1) $M \vDash \phi(a_i, b, c)$;
(2) $\forall j < i, M \vDash \neg\phi(a_i, b_j, c)$;
(3) $M \vDash \neg\phi(a, b_i, c)$; and
(4) $\forall j \geq i, M \vDash \phi(a_i, b_j, c)$.

Note that (2) and (4) give the weak order property.

Construction: Suppose we have constructed $\{a_j, b_j : j < i\}$ so far. Since $a\,{}^t\!\underset{C}{\overset{M}{\perp}} b$,

$$\phi(x,b,c) \wedge \bigwedge_{j<i} \neg\phi(x, b_j, c)$$

is realized by some $a_i \in C$. In particular, this means

$$M \vDash \phi(a_i,b,c) \wedge \bigwedge_{j<i} \neg\phi(a_i, b_j, c)$$

Similarly, $b\,{}^g\!\underset{C}{\overset{M'}{\perp}} a'$, so

$$\neg\phi(a',x,c) \wedge \bigwedge_{j\leq i} \phi(a_j, x, c)$$

is realized by some $b_i \in C$. This gives

$$M \vDash \neg\phi(a',b_i,c) \wedge \bigwedge_{j\leq i} \phi(a_j, b_i, c)$$

†

Note that "$M \vDash$" and "$M' \vDash$" are interchangeable when the parameters lie in their intersection, which includes Cb.

Proposition 8 ([BG].8.2t)**.** *$^t\!\downarrow$ satisfies Extension over models.*

Proof: Suppose $A^t \underset{M}{\overset{N}{\downarrow}} M$ and $M \subset B \subset N$. Because of this nonforking, there is an index set I and an ultrafiler U on I such that, if $h : M \to \Pi^\Gamma M/U$ is the ultrapower map, then $h(tp(A/M; N))$ is realized in $\Pi^\Gamma M/U$. Call this realization A'. Let h^+ be an isomorphism with range $\Pi^\Gamma N/U$ that contains h and set $M^{\Gamma,U} = (h^+)^{-1}(\Pi^\Gamma M/U)$ and $N^{\Gamma,U}$ similarly. Now we claim $M^{\Gamma,U\,g} \underset{M}{\overset{N^{\Gamma,U}}{\downarrow}} B$. Let $a \in M^{\Gamma,U}, b \in B, c \in M$ and $\phi(x, y, z) \in \mathcal{F}_K$. Set $[f]_U = (h^+)^{-1}(a)$. Then

$$\begin{array}{rcl} N^{\Gamma,U} & \vDash & \phi(a, b, c) \\ \Pi^\Gamma N/U & \vDash & \phi([f]_U, [i \mapsto b]_U, [i \mapsto c]_U) \\ \{i \in I & : & N \vDash \phi(f(i), b, c)\} \in U \end{array}$$

Take i_0 from this U-large set. Since $[f]_U \in \Pi^\Gamma M/U$, we have $f(i) \in M$ and satisfies $\phi(x, b, c)$. Thus, by Monotonicity, $(h^+)^{-1}(A') \underset{M}{\overset{g}{\downarrow}} b$. †

Proposition 9 ([BG].4.1t)**.** (1) *If $^t\!\downarrow$ has Existence and Extension, then it has Symmetry.*
(2) *If $^t\!\downarrow$ has Symmetry, then it has Uniqueness.*

Proof: We want to show that $A^t \underset{M}{\overset{N}{\downarrow}} B$ iff $B^t \underset{M}{\overset{N}{\downarrow}} A$; it is enough to show this for finite a and b. Suppose $a^t \underset{M}{\overset{N}{\downarrow}} b$. Then, by Existence and Extension, there is some $b' \in N'$ such that $tp_\mathcal{F}(b'/M; N') = tp_\mathcal{F}(b/M; N)$ and $b'^t \underset{M}{\overset{N'}{\downarrow}} a$. Thus, we have $tp_\mathcal{F}(ab'/M; N') = tp_\mathcal{F}(ab/M; N)$ by Proposition 7. By Invariance, $b^t \underset{M}{\overset{N}{\downarrow}} a$.

Now we want to show Uniqueness. Let $tp_\mathcal{F}(A/M; N) = tp_\mathcal{F}(A'/M; N')$, $A^t \underset{M}{\overset{N}{\downarrow}} B$, and $A'^t \underset{M}{\overset{N'}{\downarrow}} B$. By Symmetry, we have $B^t \underset{M}{\overset{N'}{\downarrow}} A'$. By Proposition 7, we have $tp_\mathcal{F}(AB/M; N) = tp_\mathcal{F}(A'B/M; N')$.

Theorem 10 ([BG].5.1t)**.** *If K is an averageable class with amalgamation that doesn't have the weak order property and $\mathcal{F}$ is closed under negation, conjunction,* *and existential quantification, then* $^t\!\downarrow$ *is an independence relation in the sense of [BG].*

Note that the closure under existential quantification is used to me Existence hold and means that $\mathcal{F}$ is likely all first-order formulas. An alternate hypothesis is that all models are $\aleph_0$ t-saturated over $\mathcal{F}$.

Proof: Existence holds because strong substructre is $\mathcal{F}$-elementary substructure. Then the rest follows from the above propositions. †

References

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