TAMENESS, POWERFUL IMAGES, AND LARGE CARDINALS

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Abstract. We provide comprehensive, level-by-level characterizations of large cardinals, in the range from weakly compact to strongly compact, by closure properties of powerful images of accessible functors. In the process, we show that these properties are also equivalent to various forms of tameness for abstract elementary classes. This systematizes and extends results of [BU17], [BTR16], [Lie18], and [LR16].

1. Introduction

Recent years have seen the rapid development of a literature concerning the implications of set-theoretic assumptions—and, in particular, large cardinal axioms—for the structure of certain useful classes of categories. This is particularly true for *accessible categories*, a category-theoretic rendering of the classes of models axiomatizable in infinitary logics (we recall all of the necessary terminology in Section 2 below). In fact, following [BTR16], [BU17], and [Lie18], we will exhibit equivalent characterizations of large cardinals in terms of, on the one hand, clean and practical closure conditions on the images of well-behaved functors between accessible categories, and, on the other hand, good behavior (specifically, *tameness*) of types in abstract model theory.

While the extent to which the structure of accessible categories depends on set theory has come as something of a surprise, it has always been clear that they occupy a complicated place between pure, i.e. abstract, category theory and ensemblist mathematics. Although there is no assumption of, or recourse to, underlying sets of objects in the definitions, constructions, and basic theory of accessible categories, infinite cardinals abound, both as indices of accessibility ("Let $\mathcal{K}$ be a κ-accessible category…") and as sizes (or, rather, *presentability ranks*) of objects. One might hope that these cardinals could be treated simply as an ordered family of indices, avoiding their more delicate arithmetical and combinatorial properties, but this has never been a possibility: per the foundational text of Makkai and Paré, for every μ and λ, every λ-accessible category is μ-accessible just in case μ is *sharply larger than* λ, [MP89, 2.3.1], a relation that reduces to cardinal arithmetic ([LRV19, 2.5], and Remark 2.5 below)[1]. That is, the *accessibility spectrum* of a general accessible category is sensitive to cardinal arithmetic, and simplifies considerably under, e.g. instances of SCH ([LRV19, 2.5, 2.7]). Similar phenomena are observed in connection with the *existence spectrum* of an accessible category $\mathcal{K}$, i.e. those cardinals κ such that it contains an object of presentability rank κ. The fact that any sufficiently large object has presentability rank a successor cardinal follows, for example, from GCH ([BR12, 2.3(5)]) or, again, instances of SCH ([LRV19, 3.11], [LRV, 5.5]).

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[1]We note that in *locally presentable categories*—special accessible categories with arbitrary (co)limits, corresponding to models of infinitary *limit theories*—such sensitivities tend to disappear, barring extreme cases along the lines of Vopěnka's Principle, cf. [AR94, §6.B]

As SCH holds above a sufficiently strongly compact cardinal (see [Jec03, 20.8]) results of this form also arise as consequences of a large cardinal assumption.

Following, and ultimately systematizing, results of [MP89], [BTR16], and [Lie18], we here concern ourselves with a more straightforwardly category-theoretic family of consequences of large cardinals: the closure properties of the images of *accessible functors*, i.e functors between accessible categories that preserve sufficiently directed colimits. The historical question motivating this line of inquiry is refreshingly simple: consider the category **Ab** consisting of abelian groups and group homomorphisms, and the full subcategory consisting of free abelian groups, **FrAb**. While **Ab** is finitely accessible (it has all directed colimits—that is, direct limits—and any abelian group is a directed colimit of finitely presented ones), things are much more complicated when it comes to **FrAb**. Under V=L, **FrAb** is not closed under κ-directed colimits in **Ab** for any uncountable regular κ: one can always construct a group G of size κ that is indecomposable—so, in particular, non-free—all of whose $< \kappa$-generated subgroups are free ([EM77], building on [She74]). Assuming a strongly compact cardinal κ, on the other hand, this problem disappears: per [EM90, 3.10], any abelian group whose $< \kappa$-generated subgroups are free is free, and **FrAb** is closed under κ-directed colimits in **Ab**. That is, **FrAb** is κ*-accessibly embedded* in **Ab**. To obtain this phenomenon at small cardinals (although still using large cardinals), see [MS94].

As a first step toward generalizing this result, note that **FrAb** is precisely the image of the free functor $F : \mathbf{Set} \to \mathbf{Ab}$, which takes a set to the free abelian group on its elements. In fact, **FrAb** is the *powerful image* of this functor, i.e. the closure of the image of F under subobjects in **Ab**: since any subgroup of a free abelian group is free abelian, **FrAb** is already closed in this sense. The functor F is finitely accessible—it preserves directed colimits—and, as we have just seen, closure of its powerful image under colimits in the codomain **Ab** is highly sensitive to set theory. One might then be inclined to ask, for example, whether, and to what extent, closure properties of the powerful images of general accessible functors are similarly conditioned by the existence of large cardinals or other set-theoretic hypotheses. The cornerstone result in this area is [MP89, 5.5.1]: if κ is strongly compact, and $F : \mathcal{K} \to \mathcal{L}$ is an accessible functor below κ, then the powerful image of F is κ-accessibly embedded in $\mathcal{L}$.[2] Careful analysis of the proof of this theorem in [BTR16] led not only to the realization that κ need only be *almost* strongly compact, but also a clearer parsing of the connection between the properties of κ as a large cardinal and the closure of the powerful image under colimits of a given shape. This allows one to link weaker compactness notions with weaker closure conditions—see [Lie18], and Section 3 below.

This is, however, just the first link in a chain of implications connecting large cardinals, category theory, and abstract model theory. The last of the three enters the picture via [Bon14] and [LR16]. The centerpiece result of the former is that if κ is a strongly compact cardinal, the Galois types in any *abstract elementary class* (or *AEC*) essentially below κ are $< \kappa$-tame; that is, they are completely determined by their restrictions to $< \kappa$-sized submodels of their domains [Bon14, 4.5]. The latter subsequently derived the same result, but by different means—one can characterize equivalence of Galois types via the powerful image of an accessible functor, in which case $< \kappa$-tameness corresponds precisely to κ-accessible embeddability of this powerful image. By [MP89, 5.5.1], as we have seen, κ-accessible embeddability, too, follows from strong compactness of κ.

[2]In fact, the standard gloss of this result is that "the powerful image is κ-accessible," or perhaps "the powerful image is κ-accessible and κ-accessibly embedded" ([BTR16]), but as we highlight in Theorem 3.3, (pre)accessibility of the powerful image holds without any assumption beyond ZFC. We therefore zero in on the closure condition on the powerful image, which genuinely depends on the large cardinal type of κ, rather than the structure of the powerful image as a category in its own right.

This chain of implications can be tightened to form an equivalence—as noted above, [BTR16, 3.2] shows that any accessible functor below an almost strongly compact cardinal κ has powerful image κ-accessibly embedded in its codomain. Moreover, we have seen that this in turn is enough to give $< \kappa$-tameness of AECs below κ. The loop is closed in [BU17] (building on [She]), where the authors build a specific AEC $\mathcal{K}$ that, subject to certain technical conditions, allows one to infer an instance of compactness from an instance of tameness of $\mathcal{K}$. So we are left with an equivalence:

Theorem 1.1 ([BU17, Corollary 4.14])**.** *Let κ be an infinite cardinal with $\mu^\omega < \kappa$ for all $\mu < \kappa$. The following are equivalent:*

1. *κ is almost strongly compact.*
2. *The powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below κ is κ-accessibly embedded in $\mathcal{L}$.*
3. *Every AEC essentially $\mathcal{K}$ below κ is $< \kappa$-tame.*

There is a great deal more to be said, however. In conjunction with [Bon14], [BU17] gives a much broader and subtler array of equivalences between gradations of compactness and gradations of tameness. One can also fine-tune the argument of [BTR16] to match these finer gradations. As noted in [Lie18], *almost measurability* of a cardinal κ implies the closure of powerful images under colimits of κ-chains, which implies $< \kappa$-locality of types in AECs below κ. Again, subject to technical assumptions, [BU17] allows us to infer almost measurability from locality, yielding another equivalence along similar lines. Here we focus on a grading of logical compactness by two parameters—(δ, θ)-*compactness*, where δ relates to the infinitary character of the language, $\mathbb{L}_{\delta,\delta}$, and θ is the cardinality of admissible theories—and give level-by-level equivalences of the same essential kind.

We give a brief review of the terminology involved in Section 2, including the precise definition of (δ, θ)-compactness, Definition 2.1. Additional background on accessible categories can be found in [MP89] and [AR94]; for AECs and tameness, see e.g. [Bal09]; for links between large cardinals and abstract model theory, see [BU17] and [Bon14]; [BTR16], [BU17], and [Lie18] provide useful context for the broader project. Section 3 derives a particular closure condition for powerful images from (δ, θ)-compactness, and in Section 4 we derive a suitable form of tameness from this closure condition. In Section 5, we bring the results of [BU17] into play, closing the loop, and obtaining the promised equivalence.

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2. Preliminaries

The large cardinals that we consider here are all related to the compactness of various infinitary logics, and provide a grading of large cardinals in the range from weakly compact to strongly compact cardinals. We note that there has been an incredible proliferation of weak forms of compactness. We insist here on the logical formulations, as these are easily motivated, correspond clearly to category-theoretic closure conditions, and tend to overlap nicely—at least on a global level—with the other existing formulations (see [Hay19] for more on the local behavior). Recall that $\mathbb{L}_{\kappa,\lambda}$ refers to the logic that allows conjunction and disjunction of $< \kappa$-many formula and quantification over $< \lambda$-many elements of the universe. A theory $T \subset \mathbb{L}_{\kappa,\lambda}$ is called *μ-satisfiable* if every $< \mu$-sized subset has a model.

As motivation for the definition that follows, we recall that finitary first order logic is compact: for any theory T in a finitary language $\mathbb{L}_{\omega,\omega}$, if T is finitely satisfiable—all of its finite subsets are satisfiable—then T is satisfiable. Whether this is true in the infinitary case is independent of ZFC: for an uncountable cardinal κ, any $<\kappa$-satisfiable theory T in an infinitary language $\mathbb{L}_{\kappa\kappa}$ is satisfiable just in case κ is a *strongly compact cardinal*. Weakly compact cardinals are defined similarly, but with a restriction on the size of the theories involved—$<\kappa$-satisfiability implies satisfiability only for theories in $\mathbb{L}_{\kappa\kappa}$ of cardinality κ. In the latter definition, it is clear that there are, morally speaking, three parameters to play with, namely the infinitary character of the language, the degree of partial satisfiability, and the cardinality of the theories for which the conclusion holds. This leads us to define:

Definition 2.1. *Let κ be an inaccessible cardinal. We say that κ is* (δ,θ)-compact*, $\delta \leq \kappa \leq \theta$, if every $<\kappa$-satisfiable theory of size θ in $\mathbb{L}_{\delta,\delta}$ is satisfiable. We say κ is $(\delta,<\theta)$-compact if it is (δ,θ')-compact for all $\theta' < \theta$, and (δ,∞)-compact if it is (δ,θ)-compact for all θ.*

We note that this encompasses a number of standard notions as special cases:

(1) κ is strongly compact if and only if it is (κ,∞)-compact.
(2) κ is almost strongly compact if and only if it is (δ,∞)-compact for every $\delta < \kappa$.
(3) κ is weakly compact if and only if it is (κ,κ)-compact.

Remark 2.2.
(1) *Although we have defined compactness in terms of $\mathbb{L}_{\delta,\delta}$, we would get the same strength if it were defined in terms of $\mathbb{L}_{\delta,\omega}$.*
(2) *The 'almost' adjective was introduced in* [BM14] *and indicates that the strong compactness is only guaranteed to exist cofinally in κ.*

We recall the following terminology related to accessible categories, and refer readers to [AR94] and [MP89] for further details:

Definition 2.3. *Let λ be a regular cardinal.*

(1) *A diagram $D: I \to \mathcal{K}$ in a category $\mathcal{K}$ is* λ-directed *if the underlying poset I is λ-directed, i.e. any $J \subset I$ with $|J| < \lambda$ has an upper bound in I.*
(2) *An object M in a category $\mathcal{K}$ is* λ-presentable *if the associated hom-functor*

$$\mathrm{Hom}_{\mathcal{K}}(M,-) : \mathcal{K} \to \mathbf{Set}$$

preserves λ-directed colimits.
(3) *A category $\mathcal{K}$ is λ-preaccessible if it contains a set $\mathcal{A}$ of λ-presentable objects such that any object of $\mathcal{K}$ is a λ-directed colimit of objects in $\mathcal{A}$.*
(4) *A category $\mathcal{K}$ is λ-accessible if it is λ-preaccessible and has all λ-directed colimits. We say $\mathcal{K}$ is accessible if it is λ-accessible for some λ.*

Definition 2.4. In an accessible category $\mathcal{K}$, every object M is λ-presentable for some λ. We define the *presentability rank* of M to be the least such λ.

Remark 2.5. Recall that accessibility (and preaccessibility) do not pass upward in an entirely straightforward fashion: while a λ-accessible category may be accessible in all regular cardinals $\mu \geq \lambda$ (i.e. it may be *well accessible*, [BR12, 2.1]) in nice special cases, it will at least be accessible in a proper class of such μ. In particular, any λ-accessible category is μ-accessible, $\mu > \lambda$ regular, if μ is *sharply greater* than λ, denoted $\mu \triangleright \lambda$. This sharp inequality relation, defined in [MP89, 2.3.1], admits a number of equivalent characterizations (see [AR94, 2.11]), e.g. $\mu \triangleright \lambda$ if and only if any subset of cardinality μ in a λ-directed poset can be completed to a λ-directed set of cardinality μ.

We note that $\triangleleft$ is strictly stronger than $<$: by [AR94, 2.13(8)], for example, it is not the case that $\aleph_1 \triangleleft \aleph_{\omega+1}$.

We note that this relation admits a clean characterization in terms of cardinal arithmetic. Recall:

Definition 2.6. *Let λ be a cardinal. We say that a cardinal μ is* λ-closed *if $\theta^{<\lambda} < \mu$ for all $\theta < \mu$.*

The following is [LRV19, 2.5]:

Fact 2.7. *Let μ and λ be regular cardinals, $\mu > \lambda$. If μ is λ-closed, $\mu \triangleright \lambda$; when $\mu > 2^{<\lambda}$, the converse holds.*

Definition 2.8. *Let λ be a regular cardinal.*

(1) *We say that a functor $F : \mathcal{K} \to \mathcal{L}$ is λ-accessible if $\mathcal{K}$ and $\mathcal{L}$ are λ-accessible, and F preserves λ-directed colimits.*
(2) *Given a subcategory $\mathcal{K}$ of a category $\mathcal{L}$, we say $\mathcal{K}$ is λ-accessibly embedded in $\mathcal{L}$ if it is full and $\mathcal{K}$ is closed under λ-directed colimits in $\mathcal{L}$.*

The central mission of the current paper, following [BTR16] and [Lie18], is to determine how close the *powerful image* of an accessible functor $F : \mathcal{K} \to \mathcal{L}$ comes to being accessibly embedded in $\mathcal{L}$, provided we work below a large cardinal κ. Recall:

Definition 2.9. *Let $F : \mathcal{K} \to \mathcal{L}$ be a functor.*

(1) *The* full image *of F is the full subcategory of $\mathcal{L}$ on objects FA, $A \in \mathcal{K}$. For brevity, we denote the full image of F by $\mathfrak{F}(F)$.*
(2) *The* powerful image *of F, denoted $\mathfrak{P}(F)$, is the closure of $\mathfrak{F}(F)$ under $\mathcal{L}$-subobjects[3]: for any M in $\mathcal{L}$, if there is an $\mathcal{L}$-monomorphism $M \to N \in \mathfrak{F}(F)$, M is in $\mathfrak{P}(F)$.*

As mentioned above, we will be concerned with accessible functors *below* particular large cardinals, a notion which we now make precise through a series of definitions:

Remark 2.10. We recall an important parameter, $\mu_{\mathcal{K},\lambda}$, associated with a λ-accessible category $\mathcal{K}$ ([BTR16, 3.1]). We proceed as follows:

(1) For any cardinal β and regular cardinal λ, define $\gamma_{\beta,\lambda}$ to be the least cardinal γ with $\gamma \geq \beta$ and $\gamma \trianglerighteq \lambda$. Define $\mu_{\beta,\lambda}$, in turn, by
$$\mu_{\beta,\lambda} = (\gamma_{\beta,\lambda}^{<\gamma_{\beta,\lambda}})^+.$$
(2) If $\mathcal{K}$ is a λ-accessible category, let $\mu_{\mathcal{K},\lambda}$ be $\mu_{\beta,\lambda}$, where β is the cardinality of $\mathbf{Pres}_\lambda(\mathcal{K})$, i.e. a full subcategory on a set of representatives of the isomorphism classes of λ-presentable objects.
(3) As we have seen, any accessible category $\mathcal{K}$ is λ-accessible for many λ: we write $\mu_{\mathcal{K}}$ to mean $\mu_{\mathcal{K},\lambda}$, where λ is the least such cardinal.

By design, $\mu_{\mathcal{K},\lambda} \trianglerighteq \gamma_{\mathcal{K},\lambda} \trianglerighteq \lambda$. As we will see in Section 3, this parameter gives a bound on the linguistic resources needed to describe $\mathcal{K}$ as a well-behaved class of structures.

We define a related parameter associated with each accessible functor:

[3]We note that it is sometimes useful to take the λ-*pure powerful image* of F, $\mathfrak{P}_\lambda(F)$, namely the closure of $\mathfrak{F}(F)$ under λ-*pure* (mono)morphisms. Indeed, [BTR16] focuses on this notion, and treats the powerful image as secondary (e.g. [BTR16, 3.4]). We take the opposite—less technically demanding—approach, indicating in Remark 3.4 how our argument can be easily modified to yield analogous results for λ-pure powerful images.

Definition 2.11. *Let $F : \mathcal{K} \to \mathcal{L}$ be an accessible functor, and let λ be the least cardinal such that F is λ-accessible. Define μ_F to be the least regular cardinal $\mu \geq \mu_{\mathcal{L},\lambda}$ such that F preserves μ-presentable objects; that is, if M is μ-presentable in $\mathcal{K}$, $F(M)$ is μ-presentable in $\mathcal{L}$.*

Definition 2.12. *Let κ be a cardinal. We say that an accessible functor $F : \mathcal{K} \to \mathcal{L}$ is* below κ *if $\mu_F < \kappa$. We say that a particular accessible category $\mathcal{K}$ is* below κ *if $\mu_{\mathcal{K}} < \kappa$.*

We also review a few of the ideas we need in connection with abstract elementary classes (AECs). First defined in [She87], AECs are a semantic (or, if you like, category-theoretic) abstraction of well-behaved nonelementary classes of models and embeddings, such as those axiomatizable in $\mathbb{L}_{\lambda,\omega}$. Naturally, they also generalize elementary classes from finitary first-order logic $\mathbb{L}_{\omega,\omega}$. Crucially, AECs retain, in the structure of their class of designated embeddings, certain essential properties of these more elementary cousins, particularly those proved without any appeal to compactness: downward Löwenheim-Skolem, the Tarski-Vaught chain, etc. For our purposes, it is enough, perhaps, to recall that an AEC $\mathcal{K}$ has arbitrary directed colimits of $\mathcal{K}$-embeddings, and that it has an associated Löwenheim-Skolem number $LS(\mathcal{K})$ such that any object in $\mathcal{K}$ is a $LS(\mathcal{K})^+$-directed colimit of $LS(\mathcal{K})^+$-presentable objects. Here the presentability rank of an object $M \in \mathcal{K}$ is precisely $|UM|^+$, i.e. the successor of the cardinality of the underlying set of M.

Notation 2.13. Whenever we work in an AEC (or, indeed, any concrete category) $\mathcal{K}$, we denote by U the functor $U : \mathcal{K} \to \mathbf{Set}$ that sends each object to its underlying set.

In AECs, the syntactic types familiar from first-order model theory are replaced by *Galois* (or *orbital*) *types*, a notion that makes sense even in a general concrete category. We adopt the more category-theoretic formulation introduced in [LR16], where we identify types in a concrete category $\mathcal{K}$ with equivalence classes of pairs of the form (f, a), where $f : M \to N$ and $a \in UN$. In particular, we say that pairs (f_1, a_1) and (f_2, a_2) are equivalent if the pointed span

$$a_1 \in N_1 \overset{f_1}{\leftarrow} M \overset{f_2}{\rightarrow} N_2 \ni a_2$$

can be extended to a commutative square

$$\begin{array}{ccc} N_1 & \xrightarrow{g_1} & N \\ \uparrow{\scriptstyle f_1} & & \uparrow{\scriptstyle g_2} \\ M & \xrightarrow[f_2]{} & N_2 \end{array}$$

with $g_1(a_1) = g_2(a_2)$. Transitivity of this relation is not automatic: one could simply take its transitive closure, but this is somewhat unwieldy from a technical perspective. If, on the other hand, we assume that $\mathcal{K}$ has the amalgamation property, i.e. any span $N_1 \leftarrow M \to N_2$ can be completed to a commutative square, this relation is already transitive. For the sake of simplicity, we adopt the latter approach.[4]

The notion of *tameness* of Galois types in a concrete category $\mathcal{K}$—essentially the requirement that equivalence of of pairs (f_1, a_1) and (f_2, a_2) over any M is determined by restrictions to subobjects of M of some fixed small size—was first isolated in [GV06], and has come to play an important role in the development of the classification theory of AECs. We consider several parameterizations of this notion, again presenting a category-theoretic formulation inspired by [LR16, 5.1].

Definition 2.14. *Let $\mathcal{K}$ be a concrete category, and let $\kappa \leq \theta$.*

[4]In fact, the arguments of this paper will also go through in the former, more technical, case, with a slight modification to Definition 2.14, and a bit of extra bookkeeping.

(1) *We say that $\mathcal{K}$ is* $(<\kappa,\theta)$-tame *if the following holds for every θ^+-presentable object M: given any (f_1, a_1) and (f_2, a_2) over M, if $(f_1\chi, a_1)$ is equivalent to $(f_2\chi, a_2)$ for all $\chi : X \to M$ with X κ-presentable, then (f_1, a_1) and (f_2, a_2) are equivalent.*
(2) *We say $\mathcal{K}$ is* $<\kappa$-tame *if it is $(<\kappa,\theta)$-tame for all $\theta \geq \kappa$.*

In light of the correspondence between cardinality and presentability in an AEC $\mathcal{K}$ (see the comment preceding 2.13 above), these clearly reduce to the standard definitions in that context, cf. [Bal09, 11.6].

3. Compactness and powerful images

We are now in a position to state the main result connecting (δ,θ)-compactness to the closure properties of powerful images. First, though, we establish the broader context for this result, encompassing [BTR16], [Lie18], and, more concretely, the free abelian group example considered in the introduction.

In connection with the latter, we note again that the free functor

$$F : \mathbf{Set} \to \mathbf{Ab}$$

is finitely accessible (i.e. ω-accessible), and that $\mathfrak{P}(F) = \mathfrak{F}(F) = \mathbf{FrAb}$, the full subcategory of $\mathbf{Ab}$ in the free abelian groups. We have the following:

Remark 3.1. Let $F : \mathbf{Set} \to \mathbf{Ab}$ be as above, κ an infinite cardinal.

(1) $\mathbf{FrAb}$ is κ-preaccessible in $\mathbf{Ab}$, i.e. any $G \in \mathbf{FrAb}$ is a κ-directed colimit in $\mathbf{Ab}$ of $<\kappa$-presentable objects. In particular, G is a κ-directed colimit of its $<\kappa$-presented subgroups, which, being subgroups of a free abelian group, must be free abelian.
(2) If κ is strongly compact, $\mathbf{FrAb}$ is closed under κ-directed colimits in $\mathbf{Ab}$, i.e. it is κ-accessibly embedded in $\mathbf{Ab}$. Notice that $\kappa \rhd \omega$ (this holds, in fact, for any uncountable regular cardinal), so $\mathbf{Ab}$ is κ-accessible. By (1), and the fact that $\mathbf{FrAb}$ is κ-accessibly embedded in $\mathbf{Ab}$, it follows that $\mathbf{FrAb}$ is κ-accessible.

As we noted in the introduction, the closure in Remark 3.1(2) genuinely depends on the large cardinal character of κ—if, for example, we take the extreme position that $V = L$, we can arrange a counterexample to closure under κ-directed colimits for any κ. The cumulative result of the generalizations of this example—[MP89, 5.5.1], [Lie18, 3.3], and particularly the proof of [BTR16, 3.2]—is a clear suggestion that large cardinal properties translate cleanly to closure conditions on powerful images of accessible functors. We summarize those results here:

Fact 3.2. Let κ be an infinite cardinal, $F : \mathcal{K} \to \mathcal{L}$ an accessible functor below κ.

(1) The powerful image of F, $\mathfrak{P}(F)$, is κ-preaccessible in $\mathcal{L}$: any object M in $\mathfrak{P}(F)$ is a κ-directed colimit of κ-presentable objects in $\mathcal{L}$ that lie in $\mathfrak{P}(F)$.
(2) If κ is an almost [measurable/strongly compact] cardinal, $\mathfrak{P}(F)$ is closed under [colimits of κ-chains/κ-directed colimits] in $\mathcal{L}$.

For the remainder of this section, we will use the machinery of the aforementioned papers to determine the closure conditions corresponding to (δ,θ)-compactness of κ.

Theorem 3.3. *Let κ be (δ,θ)-compact. For any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below δ, $\mathfrak{P}$ is*

(1) *κ-preaccessible in $\mathcal{L}$, and*

(2) *closed in $\mathcal{L}$ under θ^+-small κ-directed colimits of κ-presentable objects.*

Proof. We adhere closely to the argument for [BTR16, 3.2] (although we consider the powerful, rather than the λ-pure powerful, image, and we argue via logical compactness rather than ultraproducts—see Remark 3.4). In fact, we will condense the first steps of the aforementioned argument, in which the powerful image of the functor F—say, for definiteness, that it is λ-accessible—is encoded first as the full image of the forgetful functor

$$H : \mathcal{M} \to \mathcal{L}$$

where $\mathcal{M}$ is the category of monomorphisms $L \to FK$, $K \in \mathcal{K}$; or, to be more precise, $\mathcal{M}$ is the pullback of the comma category $(Id \downarrow F)$ in $\mathcal{L}^{\to}$ along the inclusion $\mathbf{Mono}(\mathcal{L}) \hookrightarrow \mathcal{L}^{\to}$, $\mathcal{L}^{\to}$ the usual arrow category of $\mathcal{L}$. From the pullback characterization, the category $\mathcal{M}$ is μ_F-accessible, and H preserves λ-directed colimits.

We now use the syntactic characterizations of accessible categories to simplify the problem further. In short, the categories $\mathcal{L}$ and $\mathcal{M}$ can be fully embedded in categories of many-sorted finitary structures $\mathbf{Str}(\Sigma_{\mathcal{L}})$ and $\mathbf{Str}(\Sigma_{\mathcal{M}})$, respectively, and $\mathcal{M}$ can be identified with a subcategory $\mathrm{Mod}(T)$ of $\mathbf{Str}(\Sigma_{\mathcal{M}})$ consisting of models of a $\mathbb{L}_{\mu_F,\mu_F}(\Sigma_{\mathcal{M}})$-theory T. This can also be considered as a theory in signature $\Sigma = \Sigma_{\mathcal{L}} \coprod \Sigma_{\mathcal{M}}$, with $E : \mathrm{Mod}(T) \to \mathbf{Str}(\Sigma)$ the corresponding embedding. Consider the reduct functor $R : \mathrm{Mod}(T) \xrightarrow{E} \mathbf{Str}(\Sigma) \to \mathbf{Str}(\Sigma_{\mathcal{L}})$. The full image of R, which is precisely $\mathbf{Red}_{\Sigma,\Sigma_{\mathcal{L}}}(T)$, the subcategory of reducts of models of T to $\Sigma_{\mathcal{L}}$, is equivalent to the full image of H.

Notice that the functor R preserves μ_F-directed colimits and μ_F-presentable objects. Since κ is, in particular, inaccessible, and since $\mu_F < \delta \leq \kappa$, it follows that $\mu_F \triangleleft \kappa$. By design, then, R preserves κ-directed colimits and κ-presentable objects. This means, in particular, that every M in $\mathbf{Red}_{\Sigma,\Sigma_{\mathcal{L}}}(T)$ is a κ-directed colimit of κ-presentable structures in $\mathbf{Str}(\Sigma_{\mathcal{L}})$ that lie in $\mathbf{Red}_{\Sigma,\Sigma_{\mathcal{L}}}(T)$. That is, $\mathbf{Red}_{\Sigma,\Sigma_{\mathcal{L}}}(T)$ is κ-preaccessible in $\mathbf{Str}(\Sigma_{\mathcal{L}})$, which gives us clause (1) of the theorem.

As in [BTR16] and [Lie18]—and Fact 3.2(2)—the precise nature of the large cardinal comes into play only now, in ensuring the closure of $\mathbf{Red}_{\Sigma,\Sigma_{\mathcal{L}}}(T)$ in $\mathbf{Str}(\Sigma_{\mathcal{L}})$ under colimits of diagrams of a particular size or shape—in our case, κ-directed diagrams of size at most θ and consisting of κ-presentable objects.

To that end, let $D : I \to \mathbf{Red}_{\Sigma,\Sigma_{\mathcal{L}}}(T)$ be such a diagram, and let M be its colimit in $\mathbf{Str}(\Sigma_{\mathcal{L}})$. To show that M is in $\mathbf{Red}_{\Sigma,\Sigma_{\mathcal{L}}}(T)$, it suffices to show that M admits a monomorphism into an object of $\mathbf{Red}_{\Sigma,\Sigma_{\mathcal{L}}}(T)$. We may axiomatize monomorphisms out of M by its atomic diagram; that is, the collection of all atomic and negated atomic formulas in $L_{\lambda\lambda}(\Sigma_M)$, where Σ_M is $\Sigma_{\mathcal{L}}$ with the addition of new constant symbols c_m for each $m \in UM$. So, in fact, it suffices to exhibit a model of the theory $T' = T \cup T_M$ in the new signature Σ_M—in terms of logical complexity, this theory lives in $\mathbb{L}_{\mu_F,\mu_F}(\Sigma_M)$, hence also in $\mathbb{L}_{\delta,\delta}(\Sigma_M)$.

Note that by [MP89, 2.3.11], M is θ^+-presentable in $\mathbf{Str}(\Sigma_{\mathcal{L}})$, hence $|UM| \leq \theta$. So $|T'| \leq \theta$, and we are ideally positioned to make use of (δ, θ)-compactness of κ.

Let $\Gamma \subseteq T'$, $|\Gamma| < \kappa$. In particular, $\Gamma \subseteq T \cup T_M^\Gamma$, where $T_M^\Gamma \subseteq T_M$ is of cardinality less than κ and, by a simple counting argument, involves fewer than κ of the new constants c_m. As κ-directed colimits in $\mathbf{Str}(\Sigma_{\mathcal{L}})$ are concrete, UM is the κ-directed union of the UDi, $i \in I$. By κ-directedness, some particular UDi will contain all of the relevant interpretations c_m^M in UM of the symbols in T_M^Γ. Since Di is in $\mathbf{Red}_{\Sigma,\Sigma_{\mathcal{L}}}(T)$, and the identity (mono)morphism $id_{Di} : Di \to Di$ witnesses that $Di \models T_M^\Gamma$, we have that $Di \models \Gamma$.

By (δ, θ)-compactness of κ, then, T' has a model, and we are done. $\square$

Remark 3.4.

(1) Both [BTR16] and [Lie18] use ultrafilter characterizations of large cardinals, and carry out the final step of the argument with an ultraproduct construction.
(2) In [BTR16], the authors are concerned with the λ-pure powerful image, which requires two minor modifications. First, we define $\mathcal{M}$ in a similar fashion, but insisting the morphisms be λ-pure. Second, when attempting to construct a λ-pure morphism from M in the final step, we must use its λ-*pure diagram*, consisting of positive primitive (and negated positive primitive) formulas, see [BTR16, p. 9].

Corollary 3.5. *Let κ be (δ,θ)-compact, $\theta^{<\kappa}=\theta$, and let F be an accessible functor below δ. Then if $\mathbf{Pres}_\kappa(\mathcal{L})\subset\mathfrak{P}(F)$, $\mathbf{Pres}_{\theta^+}(\mathcal{L})\subset\mathfrak{P}(F)$.*

Proof. Since $\theta^{<\kappa}=\theta$, Fact 2.7 implies that $\kappa\triangleleft\theta^+$. The category $\mathcal{L}$ is κ-accessible ($\mu_F<\kappa$, and κ is inaccessible, so $\mu_F\triangleleft\kappa$), so by [MP89, 2.3.11], any θ^+-presentable object is a θ^+-small κ-directed colimit of κ-presentables. The conclusion then follows from Theorem 3.3. $\square$

4. Powerful images and tameness

We now recast equivalence of Galois types in terms of powerful images, which will allow us to derive tameness from closure properties of the sort derived in Theorem 3.3. The approach is just as in [LR16, 5.2], [LR17, §5], and [Lie18]. In particular, we work at the level of generality used in [Lie18, 4.8], focusing not on AECs but on *concretely accessible categories*:

Definition 4.1. *A* concretely λ-accessible category *consists of a λ-accessible category $\mathcal{K}$ and a faithful functor $U:\mathcal{K}\to\mathbf{Set}$ that is λ-accessible and preserves λ-presentable objects.*

We begin by encoding equivalence of types in the form of the powerful image of a particular accessible functor—while we give the basic outline of the argument here, the details are the same as in the corresponding tameness and locality proofs mentioned above.

Definition 4.2. *Let $\mathcal{K}$ be a concretely λ-accessible category.*

(1) *We define $\mathcal{K}^<$ to be the category of pairs of (representatives of) types (f_i,a_i), $i=0,1$, i.e. spans*

$$\begin{array}{ccc} N_0 & & \\ {\scriptstyle f_0}\uparrow & & \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

with selected elements $a_i\in UN_i$, $i=0,1$.
(2) *We denote by $\mathcal{K}^\square$ the category of squares witnessing the equivalence of such pairs, i.e. commutative squares*

$$\begin{array}{ccc} N_0 & \xrightarrow{g_0} & N \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

with selected elements $a_i\in UN_i$, $i=0,1$, and $U(g_0)(a_0)=U(g_1)(a_1)$.
(3) *Let $F:\mathcal{K}^\square\to\mathcal{K}^<$ be the obvious forgetful functor.*

Remark 4.3.

(1) The full image of F, which is itself powerful—by inspection—consists of precisely the equivalent pairs (f_1, a_1), (f_2, a_2).
(2) By [Lie18, 4.7], $\mathcal{K}^{\square}$ and $\mathcal{K}^{<}$ are both λ-accessible. Moreover, F is λ^+-accessible, and preserves ν-presentable objects for arbitrary $\nu \rhd \lambda$.
(3) Following the proof of [Lie18, 2.3], we note that if $\lambda < \delta$, δ inaccessible, one can easily verify that $\mu_{\mathcal{K}^{<}} = \mu_{\mathcal{K}} < \delta$ and, in turn, that $\mu_F < \delta$.

Theorem 4.4. *Let $\delta \leq \kappa$ be inaccessible cardinals, and θ such that $\theta^{<\kappa} = \theta$. Suppose that every accessible functor F below δ has powerful image closed under θ^+-small κ-directed colimits of κ-presentable objects. Then any concretely λ-accessible category $\mathcal{K}$, $\lambda < \delta$, is $(<\kappa, \theta)$-tame.*

Proof. Let $\mathcal{K}$ be a concretely λ-accessible category, with $\lambda < \delta$, and consider $\mathcal{K}^{\square}$, $\mathcal{K}^{<}$, and $F : \mathcal{K}^{\square} \to \mathcal{K}^{<}$, as in Definition 4.2. By Remark 4.3(3), F is below δ, so the closure hypothesis in the theorem holds for the powerful image of F: it is closed under θ^+-small κ-directed colimits of κ-presentable objects.

By inaccessibility of κ, and the fact that $\lambda < \mu_F < \delta \leq \kappa$, we must have $\lambda \lhd \kappa$. Thus $\mathcal{K}$ is a concretely κ-accessible category. Moreover, the assumption on θ ensures, following the proof of Corollary 3.5, that $\theta^+ \rhd \kappa$.

Let M be a θ^+-presentable object in $\mathcal{K}$, and consider (f_i, a_i), $i = 1, 2$, with $f_i : M \to N_i$, $a_i \in UN_i$. Suppose that $(f_1\chi, a_1)$ and $(f_2\chi, a_2)$ are equivalent for any $\chi : X \to M$, X κ-presentable. By [MP89, 2.3.11], M is the colimit of a θ^+-small κ-directed colimit of κ-presentable objects, say with cocone $(\phi_i : M_i \to M \mid i \in I)$, $|I| \leq \theta$. So, in particular, $(f_1\phi_i, a_1)$ and $(f_2\phi_i, a_2)$ are equivalent for all $i \in I$.

Hence the spans

$$\begin{array}{ccc} N_1 & & \\ \uparrow{\scriptstyle f_1\phi_i} & & \\ M_i & \xrightarrow[f_2\phi_i]{} & N_2 \end{array}$$

all belong to the powerful image of F. As the original span

$$\begin{array}{ccc} N_1 & & \\ \uparrow{\scriptstyle f_1} & & \\ M & \xrightarrow[f_2]{} & N_2 \end{array}$$

is clearly the κ-directed colimit of the θ^+-small diagram of such spans, it belongs to the powerful image as well.[5] So (f_1, a_1) and (f_2, a_2) are equivalent, meaning that $\mathcal{K}$ is $(<\kappa, \theta)$-tame. $\square$

Remark 4.5. We note that any AEC $\mathcal{K}$ is a concretely $LS(\mathcal{K})^+$-accessible category—and likewise for the more general μ-AECs of [BGL+16]—so the hypotheses of Theorem 4.4 will imply $(<\kappa, \theta)$-tameness of all AECs with $LS(\mathcal{K}) < \delta$.

[5] There is a minor technical issue here, as N_1 and N_2 need not be κ-presentable, meaning that the spans, as written, need not be κ-presentable. One can remedy this by representing N_1 and N_2 as θ^+-small κ-directed diagrams of κ-presentable objects, and collating the individual diagrams into a diagram of spans of the appropriate kind: we omit the details.

5. Characterization Theorem

We now build to the main result, Theorem 5.3, which gives the promised equivalent category- and model-theoretic characterizations of (δ, θ)-compactness. We begin by summarizing what we have proven so far:

Theorem 5.1. *Let $\delta \leq \kappa$ be inaccessible cardinals, and $\theta = \theta^{<\kappa}$. Each of the following statements implies the next:*

1. *κ is (δ, θ)-compact.*
2. *If $F : \mathcal{K} \to \mathcal{L}$ is an accessible functor below δ, $\mathfrak{P}(F)$ is closed in $\mathcal{L}$ under θ^+-small κ-directed colimits of κ-presentable objects.*
3. *Every AEC $\mathbb{K}$ with $\mathrm{LS}(\mathbb{K}) < \delta$ is $(< \kappa, \theta)$-tame.*

Proof. (1) $\implies$ (2) and (2) $\implies$ (3) are, respectively, Theorems 3.3 and 4.4 (in conjunction with Remark 4.5). We note that (1) $\implies$ (3) was proven independently as [Bon14, 4.5].

□

Note that we do not have a perfect equivalence in Theorem 5.3. The following would close the loop, but the cardinal arithmetic is just off (since $\delta^{(\theta^{<\kappa})} \geq 2^\theta > \theta$).

Fact 5.2 ([BU17, Theorem 4.9.(3)])**.** *If every AEC $\mathbb{K}$ with $\mathrm{LS}(\mathbb{K}) = \delta$ is $(< \kappa, \delta^{(\theta^{<\kappa})})$-tame, then κ is (δ^+, θ)-strongly compact, in the sense that there is a δ-complete fine ultrafilter on $P_\kappa(\lambda)$.*

At the very least, this allows us to give a nice characterization at κ-closed, strong limit cardinals. Adopting the notational convention that a class is $(\mu, < \chi)$-tame if it is (μ, χ_0)-tame for all $\mu \leq \chi_0 < \chi$, we have:

Theorem 5.3. *Let $\delta \leq \kappa$ be inaccessible cardinals, and θ be a κ-closed strong limit cardinal. The following are equivalent:*

1. *κ is logically $(\delta, < \theta)$-strong compact.*
2. *If $F : \mathcal{K} \to \mathcal{L}$ is λ-accessible, $\mu_{\mathcal{L}} < \delta$, and preserves $\mu_{\mathcal{L}}$-presentable objects, then the inclusion $\mathfrak{P}(F) \hookrightarrow \mathcal{L}$ is strongly κ-accessible and $\mathfrak{P}(F)$ is closed under $< \theta$-small κ-directed colimits of κ-presentable objects.*
3. *Any AEC $\mathbb{K}$ with $LS(\mathcal{K}) < \delta$ is $(< \kappa, < \theta)$-tame.*

Proof. Combine Theorem 5.1 with Fact 5.2. □

One final time, we note that this argument can be adapted, in straightforward fashion, to the case of $\mathfrak{P}_\lambda(F)$, the λ-pure powerful image, via the modifications suggested in Remark 3.4(2).

6. Further considerations

The essential drift of this work is that large cardinals in the range from weakly to strongly compact admit both model- and category-theoretic characterizations, involving, respectively, tameness of AECs and closure properties of the powerful images of accessible functors. Moreover, these equivalences pass to parametrizations of these notions that are intrinsically natural to each of the three disciplines. Our excellent referee has asked if these characterizations can be extended to other large cardinal principles (Woodin, supercompact, and so on).

The general theme of characterizing a wide array of large cardinals by model-theoretic compactness is explored in [Bon18, BDGM]. However, in moving beyond the cardinals discussed here, at least one of two phenomena occur: either type-omission is a crucial part of the compactness (building on a result of Benda [Ben78] for supercompacts) or the compactness involves a logic stronger than (or orthogonal to) the infinitary logics $\mathbb{L}_{\kappa,\lambda}$.

In connection with the first difficulty, we submit that tameness of Galois types is a generalization of a consequence of the first-order compactness theorem to AECs. On the other hand, type-omitting compactness (which can characterize supercompact and huge cardinals in $\mathbb{L}_{\kappa,\lambda}$) is a phenomenon that fails in first-order logic. Thus we expect a different approach would have to be taken to obtain an AEC characterization of supercompacts.

As for the second, Shelah's Presentation Theorem [She87] reveals that any AEC can be seen as a reduct of a $\mathbb{L}_{\kappa,\omega}$-axiomatizable class. This means that tameness (or AEC properties more generally) are unlikely to have such a tight connection with stronger logics as we've seen here with the infinitary logics. Thus, any AEC characterization of, e.g., strong or Woodin cardinals will have to be more subtle. The same can be said of $C^{(n)}$-extendible cardinals and, therefore, Vopenka's Principle. As accessible categories are themselves axiomatizable in infinitary logic, it seems likely that these difficulties will extend to them as well.

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