ERDŐS-RADO CLASSES

WILL BONEY

Abstract. We amalgamate two generalizations of Ramsey's Theorem–Ramsey classes and the Erdős-Rado Theorem–into the notion of a combinatorial Erdős-Rado class. These classes are closely related to Erdős-Rado classes, which are those from which we can build generalized indiscernibles and blueprints in nonelementary classes, especially Abstract Elementary Classes. We give several examples and some applications.

1. Introduction

The motivation for this paper is to amalgamate two distinct generalizations of the classic Ramsey's Theorem. Ramsey's Theorem [Ram30] says that, fixing finite n and c in advance, one can find large, finite homogeneous subsets of colorings of n-tuples with c colors, as long as the set original colored was big enough. In the well-known arrow notation[1], this can be stated as follows.

Fact 1.1 (Ramsey). *For any finite k, n, c, there is finite R such that*

$$R \to (k)^n_c$$

There are two ways for this to be generalized. The first is to coloring other classes of structures. An important observation is that coloring subsets of a given finite is the same as coloring increasing tuples of that length according to some fixed linear order, so Ramsey's Theorem can be seen as a result about coloring linear orders and finding homogeneous copies of linear orders within it. A Ramsey class $\mathcal{K}_0$ is a class of finite structures where a variant of Ramsey's Theorem: given finite c and $M, N \in \mathcal{K}_0$, there is some $M^* \in \mathcal{K}_0$ such that any coloring of the copies of N appearing in M^* by c colors gives rise to a copy of M in M^* that is homogeneous for this coloring. This is written as

$$M^* \to (M)^N_c$$

Independently, Nešetřil and Rödl [NR77] and Abramson and Harrington [AH78] showed that the class of finite, linearly ordered τ-structures is a Ramsey class when τ is a finite relational language. Since then the theory of Ramsey classes has become a productive area connecting combinatorics, dynamics, and model theory (the connection to model theory is partially explained below; a nice survey on Ramsey classes is Bodirsky [Bod15]).

In another direction, one might want to remove the restriction 'finite' in the statement of Ramsey's Theorem. Allowing the arity of the coloring (the upper exponent in the arrow relation) to be infinite would make positive results contradict the axiom of choice (see [EHMR84, Theorem 12.1]), so we focus on finite arity colorings. Ramsey's Theorem can be easily generalized to

Date: October 22, 2018.

[1]The notation

$$\alpha \to (\beta)^r_\gamma$$

means that for any coloring $c : [\alpha]^r \to \gamma$, there is $X \subset \alpha$ of type β such that $c``[X]^r$ is a single element; such an X is called *homogeneous*. Hajnal and Larson [HL10, p. 130] point out "[t]here are cases in mathematical history when a well-chosen notation can enormously enhance the development of a branch of mathematics and a case in point is the ordinary partition symbol."

$\omega \to (\omega)^n_c$ for all finite n, c. Moving to infinitely many colors and uncountable homogeneous sets, Erdős and Rado [ER56] proved the following (and, unlike finite Ramsey theory, the left-hand cardinal is known to be optimal).

Fact 1.2 (Erdős-Rado). *For any finite n and infinite λ,*

$$\beth_{n-1}(\kappa)^+ \to (\kappa^+)^n_\kappa$$

This has been generalized in many directions, including unbalanced and polarized partition relations. Excellent surveys can be found in Erdős, Hajnal, Máté, and Rado [EHMR84] and Hajnal and Larson [HL10].

We give a general framework for generalizations of the Erdős-Rado Theorem along the lines of Ramsey classes, appropriately called combinatorial Erdős-Rado classes (Definition 3.5, see later in this introduction for a discussion of Erdős-Rado classes). Roughly, a class $\mathcal{K}$ is a combinatorial Erdős-Rado class if it satisfies enough instances of $\lambda \xrightarrow{\mathcal{K}} (\kappa^+)^n_\kappa$, where this means any coloring of n-tuples from any λ-big structure in $\mathcal{K}$ with κ-many colors has a homogeneous substructure that is κ^+-big (Section 3 makes these notions of 'big' and 'homogeneous' precise). Note that we require that all n-tuples are colored, rather than coloring copies of a single structure in Ramsey classes. Many partition relations of this sort (positive and negative) already exist in the literature, and we collect the most relevant and place them in this framework in Section 3.1.

Our main interest in these results comes from model theory, specifically building generalized indiscernibles in nonelementary classes. An (order) indiscernible sequence indexed by a linear order I is a sequence $\{\mathbf{a}_i : i \in I\}$ in a structure M where the information (specifically, the type) about the elements $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}$ computed in the structure M only depends on the ordering of the indices $i_1, \dots, i_n$. Generalized indiscernibles replace the linear orders with some other index class: trees, functions spaces, etc . Generalized indiscernibles (and the related notion of generalized blueprints) appear in Shelah [She90], and we recount the definitions in Section 2.

In elementary classes (those axiomatizable in first-order logic), indiscernibles exists because of Ramsey's Theorem. Moving to more complicated index classes $\mathcal{K}$, the combinatorics necessary to build generalized indiscernibles from $\mathcal{K}$ are exactly the same as requiring that $\mathcal{K}$ be the directed colimits of a Ramsey class $\mathcal{K}_0$ (see [Sco12, Theorem 4.31]). Roughly, this means that $\mathcal{K}_0$ is a collection of structures in a finite relational language such that if one fixes $n, c < \omega$ and $N \in \mathcal{K}_0$, then for all large enough (still finite) $M \in \mathcal{K}_0$ and all colorings of n-tuples from M with c colors, there is a monochromatic copy of N. In both of these constructions, restriction to finite structures is sufficient to build indiscernibles because the compactness theorem reduces satisfiability to satisfiability of finite sets.

The study of nonelementary classes typically focuses on those axiomatizable in nice logics beyond first-order and, slightly more broadly, on Abstract Elementary Classes. Abstract Elementary Classes (introduced by Shelah [She87a]) give an axiomatic framework for a class of structures $\mathbb{K}$ and a strong substructure notion $\prec_{\mathbb{K}}$ meant to encompass a wide variety nonelementary classes. A key feature of nonelementary classes is that the lack the structure that the compactness theorem endows on elementary classes. Indeed, Lindström's Theorem [Lin69] says that no logic stronger than first-order can satisfy the classical (countable) compactness theorem and the downward Löwenheim-Skolem property. In practice, stronger logics tend to fail compactness (the cofinality quantifier logics $\mathbb{L}(Q^{\text{cof}}_\alpha)$ are a notable exception [She75]). Thus, different methods are necessary to build indiscernibles in Abstract Elementary Classes.

For order indiscernibles, this method comes by way of Morley's Omitting Types Theorem [Mor65] using the Erdős-Rado Theorem mentioned above (this is discussed in [Bal09, Chapter 4 and Appendix A]. For generalized indiscernibles, the generalization of the Erdős-Rado Theorem to combinatorial Erdős-Rado classes described above gives the desired tools. We call $\mathcal{K}$ an Erdős-Rado class if we can build $\mathcal{K}$-indiscernibles in any Abstract Elementary Class (Definition 4.1 and

Theorem 4.6). Generalized indiscernibles have occasionally seen use in nonelementary classses (for instance, [GS86], [She09, Chapter V.F], [BS12]).

The use of structural partition relations allows us to present a unified framework for generating generalized indiscernibles in nonelementary classes. This allows us to generalize Morley's result as Theorem 4.2. There is also some work in this direction in Shelah [She], and we compare them in Remark 4.7.

We also make explicit category theoretic formulations of (generalized) Ehrenfeucht-Mostowski models and indiscernible collapse. This is motivated by a statement of Morley's Omitting Types Theorem by Makkai and Paré in their work on accessible categories [MP89, Theorem 3.4.1]. Essentially, generalized blueprints correspond to nice functors, and we prove a converse to this as well (Theorem 5.5).

1.1. **Outline.** Section 2 gives the necessary preliminaries on abstract classes of structures, types, and generalized indiscernibles and blueprints. We also include a description in Section 2.3 of the examples we will consider in this paper. Section 3 gives the definition of combinatorial Erdős-Rado classes and of the structural partition relation that defines them. Section 3.1 gives several known (and a few new) examples and counterexamples of these classses. Section 4 defines Erdős-Rado classes and proves the main link between the two notions, Generalized Morley's Omitting Types Theorem 4.2. Section 5 describes several extensions and partial converses to this result, including the category theoretic perspective on blueprints. Section 6 gives three applications of this technology: stability spectra of tame AECs, indiscernible collapse in nonelementary classes, and the interpretability order. We thank Saharon Shelah for pointing out the argument for Proposition 3.9, although he suggests it is well-known.

Note that the definition of Erdős-Rado classes (Definition 4.1) does not actually depend on that of combinatorial Erdős-Rado classes (Definition 3.5) or any of Section 3. However, we give the combinatorial definitions first, as they provide the largest class of examples of Erdős-Rado classes.

1.2. **Conventions.** Throughout the paper, we deal with different classes of structures, normally referred to by K with some decoration. To aid the reader, we observe the following convention:

- the script or calligraphic K–typeset as $\mathcal{K}$–will be used as the domain or index class that we wish to build generalized indiscernibles *from.* They typically have few assumptions of model-theoretic structure on them. Erdős-Rado classes will be of this type, and linear order form the prototypical example.
- the bold K–typeset as $\mathbb{K}$–will be used as the target class that we wish that we wish to build generalized indiscernibles *in.* They will typically be well-structured in some model-theoretic sense. Elementary classes and Abstract Elementary Classes form the prototypical examples.

We also observe two important conventions with respect to types that might be missed by the model-theoretically inclined reader that skips the Preliminaries Section (see Definition 2.2):

(1) Since we never deal with types over some parameter set, we omit the domain of types through out. For example, we write $tp_{\mathcal{K}}(\mathbf{a}; I)$ for the $\mathcal{K}$-type of $\mathbf{a}$ over the empty set computed in I, rather than $tp_{\mathcal{K}}(\mathbf{a}/\emptyset; I)$; and
(2) $\mathbb{K}^\tau$ is the class of all τ-structures with τ-substructure. In particular, tp_τ is the type in this class, which turns out to be quantifier-free type.

Note that Sections 5.3 and 6.1 require more knowledge about Abstract Elementary Classes. This can be found in, e.g., Baldwin [Bal09].

2. Preliminaries

2.1. **Classes of structures and types.** We want to have a very general framework for classes of structures in a common language along with a distinguished substructure relation. Although much more general than we need, we can use the notion of an abstract class (this formalization is originally due to Grossberg). Additionally, we expect our Erdős-Rado classes to have orderings (similar to, e.g., [Bod15, Proposition 2.2] for Ramsey classes), so we introduce the notion of an ordered abstract class. An alternative would be to consider equivalence classes of types in the Stone space after modding out by permutation of the indices, but requiring an ordering seems simpler.

Note that the examples presented tend to be universal classes (and mostly relational), in which case the type is determined by the quantifier-free type in $\mathbb{L}_{\omega,\omega}$. However, we offer a more general framework because it adds little technical difficulty and offers the possibility to wider applicability. For instance, well-founded trees (Example 2.11, 3.12) are **not** a universal class.

Definition 2.1.

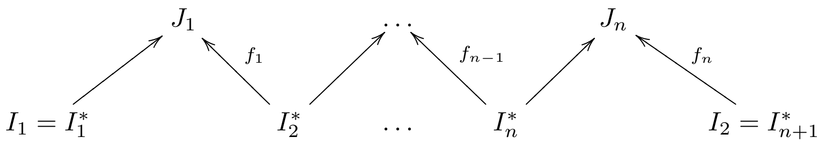
(1) $(\mathcal{K}, \leq_\mathcal{K})$ *is an* abstract class *iff there a language* $\tau = \tau(\mathcal{K})$ *such that each* $M \in \mathcal{K}$ *is a* τ*-structure,* $\leq_\mathcal{K}$ *is a partial order contained in* $\subset_\tau$*, and membership in* $\mathcal{K}$ *and* $\leq_\mathcal{K}$ *both respect isomorphism. We often refer to the class simply as* $\mathcal{K}$*.*

(2) $(\mathcal{K}, \leq_\mathcal{K})$ *is an* ordered abstract class *iff it is an ordered class with a distinguished binary relation* $<$ *in* $\tau(\mathcal{K})$ *such that* $<^I$ *is a total order of* I *for every* $I \in \mathcal{K}$*.*

We will also use the types of elements. Most of the classes we consider will not be elementary (either in axiomatization of $\mathcal{K}$ or ordering $\prec_\mathcal{K}$), so syntactic types give way to semantic notions. Specifically, we use the notion of Galois types (also called orbital types) used in the study of Abstract Elementary Classes (and originated in [She87b]). However, in most cases, this will be the same as quantifier-free types. Note that we typically drop any adjective and use 'type' or sometimes '$\mathcal{K}$-type' to refer to the following semantic definition, although we will decorate the symbol with the ambient class.

Definition 2.2. *Let* $\mathcal{K}$ *be an abstract class.*

(1) *Given* $I_1, I_2 \in \mathcal{K}$ *and* $\mathbf{a}_1 \in I_1$*,* $\mathbf{a}_2 \in I_2$*, we say that* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ have the same $\mathcal{K}$-type *iff there are* $J_1, \dots, J_n; I_1^*, \dots, I_{n+1}^* \in \mathcal{K}$*,* $\mathbf{b}_\ell \in I_\ell^*$*, and* $\mathcal{K}$*-embeddings* $f_\ell : I_{\ell+1}^* \to J_\ell$ *such that*
   (a) $I_1^* = I_1$*,* $I_{n+1}^* = I_2$*,* $\mathbf{a}_1 = \mathbf{b}_1$*, and* $\mathbf{a}_2 = \mathbf{b}_{n+1}$*;*
   (b) $I_\ell^* \leq_\mathcal{K} J_\ell$*; and*
   (c) $\mathbf{b}_\ell = f_\ell(\mathbf{b}_{\ell+1})$*.*



We write $tp_\mathcal{K}(\mathbf{a}; I)$ *to be the equivalence class*[2] *of all tuples that have the same type as* $\mathbf{a}$*. Thus, '*$tp_\mathcal{K}(\mathbf{a}_1; I_1) = tp_\mathcal{K}(\mathbf{a}_2; I_2)$*' has the same meaning as '*$\mathbf{a}_1$ *and* $\mathbf{a}_2$ *have the same type.'*

(2) $S_\mathcal{K} := \{tp_\mathcal{K}(\mathbf{a}; I) \mid \mathbf{a} \in I \in \mathcal{K}\}$ *is the* Stone space *or space of types.*

(3) *If* $\mathcal{K}$ *is an ordered abstract class, then* $S_\mathcal{K}^{inc}$ *is the subset of* $S_\mathcal{K}$ *whose realizations are in increasing order, namely,*

$$S_\mathcal{K}^{inc} := \{tp_\mathcal{K}(\mathbf{a}; I) \mid \mathbf{a} \in I \in \mathcal{K} \text{ and } a_1 < \dots < a_n\}$$

[2]This is a proper class, but we can use Scott's trick (see [Jec02, p. 65]) or some other method to only deal with sets.

(4) *Adding a subscript* $n < \omega$ *to either* $S_\mathcal{K}$ *or* $S^{inc}_\mathcal{K}$ *restricts to looking at types of* n*-tuples.*
(5) *Let* $p \in S^n_\mathcal{K}$ *be* $tp_\mathcal{K}(i_1, \dots, i_n; I)$ *and* $s \subset n$ *be* $k_1 < \cdots < k_m$ *for* $m = |s|$. *Then* $p^s := tp_\mathcal{K}(i_{k_1}, \dots, i_{k_m}; I) \in S^m_\mathcal{K}$.
(6) *If we have an ordered abstract class decorated with a superscript* $\mathcal{K}^x$*, then we often use this superscript in place of the whole class in this notation, e.g.,* $S_{\chi-or}$ *rather than* $S_{\mathcal{K}^{\chi-or}}$.

2.2. **Generalized indiscernibles and blueprints.** The following generalizes the normal theory of blueprints and Ehrenfeucht-Mostowski models began in [EM56]. These generalized notions appear in [She90, Section VII.2].

Definition 2.3. *Let* $\mathcal{K}$ *be an ordered abstract class.*

(1) *A* blueprint Φ proper for $\mathcal{K}$ *is a function* $\Phi : S^{inc}_\mathcal{K} \to S_\tau$ *for some* $\tau = \tau(\Phi)$ *that satisfies the following coherence conditions:*
   (a) *the free variables of* $\Phi(p)$ *are the free variables of* p*; and*
   (b) *given variables* $s \subset n$ *and* $p \in S^{inc,n}_\mathcal{K}$*, we have that*
$$\Phi\left(p^s\right) = \Phi(p)^s$$
   $\Upsilon^\mathcal{K}$ *is the collection of all blueprints proper for* $\mathcal{K}$.
   $\Upsilon^\mathcal{K}_\kappa$ *is the collection of all blueprints proper for* $\mathcal{K}$ *such that* $|\tau(\Phi)| \leq \kappa$.
(2) *Let* $I \in \mathcal{K}$ *and* Φ *be a blueprint proper for* $\mathcal{K}$. *Then, we can build a* $\tau(\Phi)$*-structure* $EM(I, \Phi)$ *such that, for all* $i_1 < \cdots < i_n \in I$*, we have that*
$$tp_\tau\left(i_1, \dots, i_n; EM(I, \Phi)\right) = \Phi\left(tp_\mathcal{K}(i_1, \dots, i_n; I)\right)$$
   and that every element of $EM(I, \Phi)$ *is a* $\tau(\Phi)$*-term of a sequence from* I.
   If $\tau \subset \tau(\Phi)$*, then* $EM_\tau(I, \Phi) := EM(I, \Phi) \upharpoonright \tau$.
(3) *Given an class* K *of* τ*-structures and a blueprint* Φ *with* $\tau \subset \tau(\Phi)$*, we say that* Φ is proper for $(\mathcal{K}, K)$ *iff it is proper for* $\mathcal{K}$ *and, for any* $I \in \mathcal{K}$*,* $EM_\tau(I, \Phi) \in K$.
   $\Upsilon^\mathcal{K}[K]$ *is the collection of all blueprints proper for* $(\mathcal{K}, K)$.
   $\Upsilon^\mathcal{K}_\kappa[K]$ *is the collection of all blueprints proper for* $(\mathcal{K}, K)$ *such that* $|\tau(\Phi)| \leq \kappa$.
(4) *Given an abstract class* $\mathbb{K} = (K, \prec_\mathbb{K})$ *and a blueprint* Φ *with* $\tau(\mathbb{K}) \subset \tau(\Phi)$*, we say that* Φ is proper for $(\mathcal{K}, \mathbb{K})$ *iff it is proper for* $(\mathcal{K}, K)$ *and, for any* $I \leq J \in \mathcal{K}$*,* $EM_{\tau(\mathbb{K})}(I, \Phi) \prec_\mathbb{K} EM_{\tau(\mathbb{K})}(I, \Phi)$.
   $\Upsilon^\mathcal{K}[\mathbb{K}]$ *is the collection of all blueprints proper for* $(\mathcal{K}, \mathbb{K})$.
   $\Upsilon^\mathcal{K}_\kappa[\mathbb{K}]$ *is the collection of all blueprints proper for* $(\mathcal{K}, \mathbb{K})$ *such that* $|\tau(\Phi)| \leq \kappa$.

Being proper for $\mathcal{K}$ is the same as being proper for $(\mathcal{K}, \mathbb{K}^{\tau(\Phi)})$. Note that the description in Definition 2.3.(2) uniquely describes a model, but is short on proving it's existence. However, the existence of such a model follows from standard arguments about EM models, see, e.g., [Mar02, Section 5.2]. Note that our formalism has I be the generating set for $EM(I, \Phi)$ (and later indiscernibles), rather than passing to a skeleton.

From a category-theoretic perspective, a blueprint $\Phi \in \Upsilon^\mathcal{K}[\mathbb{K}]$ induces a functor $\Phi : \mathcal{K} \to \mathbb{K}$ that is faithful, preserves colimits, and induces a natural transformation between the 'underlying set' functor of each concrete category; see Section 5.2 for more. We return to this perspective in Section 5.2 and derive a converse Theorem 5.2 of the Generalized Morley's Omitting Types Theorem 4.2.

Example 2.4.

(1) *These definitions generalize the standard notions of blueprints and Ehrenfeucht-Mostowski models when* $\mathcal{K}$ *is the class of linear orders.*
(2) *Consider a bidimensional theory like the theory* $T = Th(\oplus\mathbb{Z}(p^\infty))$ *of the direct sum of countably many copies of the Prüfer* p*-group. Any model of* T *is some infinite direct*

sum of copies of $\mathbb{Z}(p^\infty)$ *and* $\mathbb{Q}$*. So each model is given by the number of copies of each of these structures. Using standard Ehrenfeucht-Mostowski models, one could only get blueprints that either vary one dimension and not the other* **or** *make the dimensions the same.*

However, there is a generalized blueprint $\Phi \in \Upsilon^{2-or}_{\aleph_0}[T]$ *for the class of two disjoint linear orders that takes* (I, J) *to the model*

$$\bigoplus_{i \in I} \mathbb{Z}(p^\infty) \oplus \bigoplus_{j \in J} \mathbb{Q}$$

Thus, every model of T *is isomorphic to* $EM_\tau((I, J), \Phi)$ *for some* I *and* J*.*

Using generalized blueprints, we can build models with generalized indiscernibles (see Theorem 4.6 for this in action).

Definition 2.5. *Let* $\mathcal{K}$ *an ordered abstract class and* $\mathbb{K}$ *be an abstract class. Then, given* $I \in \mathcal{K}$ *and* $M \in \mathbb{K}$*, a collection* $\{\mathbf{a}_i \in {}^{<\omega}M \mid i \in I\}$ *is a* $\mathcal{K}$-indiscernible sequence *iff for every* $i_1, \dots, i_n; j_1, \dots, j_n \in I$*, if*

$$tp_\mathcal{K}(i_1, \dots, i_n; I) = tp_\mathcal{K}(j_1, \dots, j_n; I)$$

then

$$tp_\mathbb{K}(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}; M) = tp_\mathbb{K}(\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_n}; M)$$

An important fact to keep in mind is that, in nonelementary classes, not every collection of indiscernibles can be turned into a blueprint; [Bal09, Example 18.9] provides such an example. This is in contrast to first-order, where every infinite set of indiscernibles can be stretched (see [TZ12, Lemma 5.1.3]).

2.3. **Our examples.** There will be several examples that we will develop here and in Section 3.1. Here, we define the relevant classes and note the syntactic characterization of their types (normally quantifier-free). Section 3.1 explains how these classes fit within the Erdős-Rado class framework.

Example 2.6 (Linear orders)**.** $\mathcal{K}^{or}$ *is the class of linear orders with substructure, specifically in the language with a single binary relation* $<$ *that is the ordering. This is an ordered abstract class and is universal, so* $\mathcal{K}^{or}$*-type is simply quantifier-free type. This is our prototypical Erdős-Rado class.*

Example 2.7 (χ disjoint linear orders)**.** $\mathcal{K}^{\chi-or}$ *is the class of* χ *disjoint linear order. In order to make this an ordered abstract class, we say* $\bar{I} \in \mathcal{K}^{\chi-or}$ *consists of a disjoint sets* $\{I_i\}_{i<\chi}$ *and a total ordering* $<$ *such that* $i < j < \chi$ *implies that* $I_i << I_j$ *(*$X << Y$ *means that every element of* X *is below every element of* Y*). Note that if* χ *is infinite, then this is not an elementary class.*

Example 2.8 (χ-colored linear orders)**.** *We set* $\mathcal{K}^{\chi-color}$ *to be a particular class of colored linear orders.* $(I, <, P_\beta)_{\beta<\chi} \in \mathcal{K}^{\chi-color}$ *consists of a well-ordering* $(I, <)$ *such that* $P_\beta = \{i \in I :$ i *is the* $(\gamma \cdot \mu + \beta)$*th element of* I *for some* $\gamma\}$*.*

Example 2.9 (Trees of height $n < \omega$)**.** *Fix the language* $\tau_{n-tr} = (P_k, <, \prec, \wedge)_{k<n}$*. Then* $\mathcal{K}^{n-tr}$ *consists of all* τ_{n-tr}*-structures* I *such that*

- $(I, \prec)$ *is a tree of height* n*;*
- P_k *are all vertices on level* n*;*
- $<$ *is a total order of* I *coming from a lexicographic ordering of the tree; and*
- $\wedge$ *is the meet operation on this tree.*

Then $\mathcal{K}^{n-tr}$*-type is just quantifier-free type in this language.*

Example 2.10 (Trees of height ω). *$\mathcal{K}^{\omega-tr}$ are the trees of height ω formalized in the language $\tau_{\omega-tr} = \cup_{n<\omega}\tau_{n-tr}$.*

Of course, these tree examples can be continued on past height ω, but we know of no results (positive or negative) on these classes in terms of the Erdős-Rado notions (see Question **??**).

Example 2.11 (Well-founded trees). *$\mathcal{K}^{wf-tr}$ are the well-founded trees formalized in the language $\tau_{\omega-tr}$; recall a tree is well-founded iff it contains no infinite branch. Well-founded trees can be identified with decreasing sequences of ordinals.*

Example 2.12 (Convexly-ordered equivalence relations). *A convexly ordered equivalence relation is $(I, <, E)$, where E is an equivalence relation on I, $<$ is a total order, and*

$$\forall x, y, z \in I\,(xEz \wedge x < y < z \to xEy)$$

$\mathcal{K}^{ceq}$ is the collection of all such structures. These are similar to the class $\mathcal{K}^{\chi-or}$ except the χ is allowed to vary. However, the type of, e.g., singletons in different equivalence classes is the same. This will make finding type homogeneous sets for colorings more difficult.

Example 2.13 (n-multi-linear orders). *A n-multi-linear order is $(I, <_1, \ldots, <_n)$ where each $<_i$ is a linear order of I. $\mathcal{K}^{n-mlo}$ is the class of these. We take $<_1$ as the distinguished linear order to view this as an ordered abstract class.*

Example 2.14 (Ordered graphs). *$\mathcal{K}^{og}$ consists of the class of all ordered graphs.*

Example 2.15 (Colored hypergraphs). *Fix $k \leq \omega$ and a cardinalσ. $\mathcal{K}^{(k,\sigma)-hg}$ consists of all $(I, <, F)$ where $<$ is a well-ordering and $F : [A]^{<k} \to \sigma$ is a function. If $\sigma = 2$, then one can think of $\mathcal{K}^{(\alpha,2)-hg}$ as the collection of all hypergraphs with all edge arities $< k$.*

3. Structural Partition Relations and Combinatorial Erdős-Rado Classes

We will formulate a version of the normal partition relation for classes other than linear orders in Definition 3.4. This will encapsulate the idea that any coloring of n-tuples from a large structure will have a large substructure that behaves the same way to this coloring. First, we consider an example that indicates some of the difficulties and the need for new concepts, namely bigness notions (Definition 3.2) and type-homogeneity (Definition 3.3).

Example 3.1. *Let $\bar{I} = (I_0, I_1) \in \mathcal{K}^{2-or}$ (recall Example 2.7). Define a coloring $c : [\bar{I}]^2 \to 2$ based on wether the two elements are in the same partition: given $i, j \in (I_0, I_1)$, set*

$$c(\{i, j\}) = \begin{cases} 0 & i \in I_0 \iff j \in I_1 \\ 1 & \text{otherwise} \end{cases}$$

Then any $\bar{I}^* \subset \bar{I}$ that contains at least one element from one partition and two from the other will not be homogeneous for this coloring no matter what $\bar{I}$ is.

This example exposes two issues.

- First, we could take $\bar{I}^*$ to be $(\emptyset, I_1)$, which is homogeneous for this coloring. However, taking one of the partitions to be empty goes against the point of working in $\mathcal{K}^{2-or}$. So we will attach to these classes a notion of size (or bigness).
- Second, we colored the pairs using information about their type. This meant that we could place restrictions on the structure of any homogeneous subset. To allow for big homogeneous sets we will allow for the 'single color' to depend on the type of tuple.

For the first issue, we define abstractly what it means to be a bigness notion. The only requirements are a monotonicty condition and some weak degree of saturation. For each class from Subsection 2.3, we make its associated bigness notion explicit in Subsection 3.1. It seems like the bigness notions there are essentially saturation after removing the linear order, but we prefer the greater flexibility afforded by an abstract notion.

Definition 3.2. *Let $\mathcal{K}$ be an abstract class. A* bigness notion **big** *for $\mathcal{K}$ is a (definable) collection $\{\mathcal{K}^{\boldsymbol{big}}_\mu \subset \mathcal{K} \mid \mu \in Card\}$ such that*

(1) *if $\mu_1 \leq \mu_2$ and $M \leq_{\mathcal{K}} N$, then $M \in \mathcal{K}^{\boldsymbol{big}}_{\mu_1}$ implies that $N \in \mathcal{K}^{\boldsymbol{big}}_{\mu_2}$; and*
(2) *if $M \in \mathcal{K}^{\boldsymbol{big}}_{\aleph_0}$, then every type in $S_{\mathcal{K}}$ is realized in M.*

*We write '$M \in \mathcal{K}$ is μ-**big**' for '$M \in \mathcal{K}^{\boldsymbol{big}}_\mu$.' Also, we will typically only have one have one bigness notion for a given class, so we will omit it.*

Note that the omission of **big** will lead to some nonstandard notation, e. g., $\mathcal{K}^{\chi-or}_\mu$ are the μ-big elements of $\mathcal{K}^{\chi-or}$ according to the bigness notion given in Example 3.7, rather than all elements of $\mathcal{K}^{\chi-or}$ whose universe has cardinality μ. Iin particular $\mathcal{K}^{\chi-or}_\mu$ will have structures that are bigger than μ-big when used in this paper.

Turning to homogeneity, the key observation from Example 3.1 was that the types of tuples are extra information that can be used to define a coloring. In the class of linear orders, there is only one increasing type of an n-tuple, so this issue doesn't arise. In the general case, we can always use the type as information to color a tuple, so we want homogeneity to mean that the type is the *only* information that can be used to determine the color of a type.

Definition 3.3. *Let $\mathcal{K}$ be an ordered abstract class, $I \in \mathcal{K}$, and $c : [I]^n \to \kappa$. We say that $I_0 \leq_{\mathcal{K}} I$ is* type-homogeneous for c *iff the color of a tuple from I is determined by the $\mathcal{K}$-type of it listed in increasing order; that is, there is a function $c^* : S^{inc,n}_{\mathcal{K}} \to \kappa$ such that, for any distinct $i_1 < \cdots < i_n \in I_0$, we have that*

$$c(\{i_1, \ldots, i_n\}) = c^*\left(tp_{\mathcal{K}}(i_1, \ldots, i_n; I)\right)$$

In Example 3.1, the entire set $\bar{I}$ is type-homogeneous for the given coloring.

With these new concepts in hand, we can define the structural partition relation.

Definition 3.4. *Let $\mathcal{K}$ be an ordered abstract class with a bigness notion **big**. Given cardinal $\mu, \lambda, \alpha, \kappa$, we write*

$$(\lambda) \xrightarrow[\textbf{big}]{\mathcal{K}} (\mu)^\alpha_\kappa$$

*to mean that given any λ-**big** $I \in \mathcal{K}$ and coloring $c : [I]^\alpha \to \kappa$, there is a μ-**big** $I_0 \subset I$ from $\mathcal{K}$ that is type-homogeneous for c.*

If $\mathcal{K}$ is one of our examples with an associated bigness notion and is denoted $\mathcal{K}^x$, then we simply write

$$(\lambda) \xrightarrow{x} (\mu)^\alpha_\kappa$$

for $(\lambda) \xrightarrow[\textbf{big}]{\mathcal{K}} (\mu)^\alpha_\kappa$.

Since the associated bigness notion for $\mathcal{K}^{or}$ is simply cardinality, $(\lambda) \xrightarrow{or} (\mu)^\alpha_\kappa$ is the normal partition relation. In particular, positive instances of the structural partition relation are guaranteed by the Erdős-Rado Theorem, which states that $\beth_{n-1}(\kappa)^+ \xrightarrow{or} (\kappa^+)^n_\kappa$ for every cardinal κ and every $n < \omega$.

Polarized partition relations (see [EHMR84, Section III.8.7]) are similar to $\xrightarrow{\chi-or}$, but typically specify (in our language) the type of the tuple to be considered (and so are more like the Ramsey class-style partition relations).

We will list several further positive instances of structural partition relations (new and old) in Subsection 3.1.

From the structural partition relation, we can define combinatorial Erdős-Rado classes as those that satisfy structural partition relations for all inputs on the right side.

Definition 3.5. *Let $\mathcal{K}$ be an ordered abstract class with a bigness notion* ***big****. We say that $\mathcal{K}$ is a* combinatorial Erdős-Rado class *iff there is some function $F : Card \times \omega \to Card$ such that, for every $\kappa < \mu$ and $n < \omega$, we have that*

$$(F(\mu, n)) \xrightarrow[\mathbf{big}]{\mathcal{K}} (\mu)^n_\kappa$$

We refer to the function F as a witness.

3.1. **Examples and some counter-examples.** We show that the examples introduced in Section 2.3 are combinatorial Erdős-Rado classes (or mention results indicating they are not). In most cases, no claim of the optimality of the witnessing functions is made. While interesting from a combinatorial perspective, any reduction of the bounds on the order of 'finitely many power set operations' will not affect the witnesses for these classes being Erdős-Rado via an application of the Generalized Morley's Omitting Types Theorem. Note that none of these results were originally stated in the notation of Definition 3.4 (especially since that notation was originated for this paper); however, we have translated those results into this language to illustrate our notions.

Example 3.6 (Linear orders)**.** *In $\mathcal{K}^{or}$, the canonical bigness notion is just cardinality, so $I \in \mathcal{K}^{or}_\mu$ iff $|I| \geq \mu$. The classic Erdős-Rado [ER56] theorem states that, for all $n < \omega$ and κ*

$$\beth_{n-1}(\kappa)^+ \xrightarrow{or} (\kappa^+)^n_\kappa$$

Thus, $\mathcal{K}^{or}$ is a combinatorial Erdős-Rado class witnessed by $(\kappa, n) \mapsto \beth_{n-1}(\kappa)^+$.

Note that the classic results on the Sierpinski coloring show that *dense* linear orders do **not** form a combinatorial Erdős-Rado class.

Example 3.7 (χ-disjoint linear orders)**.** *As discussed in the context of Example 3.1, the canonical bigness notion for $\mathcal{K}^{\chi-or}$ says that $\bar{I}$ is μ-big iff every piece has size at least μ. Erdős, Hajnal, and Rado [EHR65] give a polarized partition relation that says, essentially, for all $n < \omega$,*

$$\beth_{n(n+1)}(\chi)^+ \xrightarrow{\chi-or} (\chi^+)^n_\chi$$

[She90, Appendix, Theorem 2.7] also provides a proof; note that the full statement is stronger and does not requiring the ordering on the disjoint parts. By adding dummy sets, this can be strengthened to show that, for all $\kappa \geq \chi$ and $n < \omega$,

$$\beth_{n(n+1)}(\kappa)^+ \xrightarrow{\chi-or} (\kappa^+)^n_\kappa$$

Thus, $\mathcal{K}^{\chi-or}$ is a combinatorial Erdős-Rado class witnessed by $(\kappa^+, n) \mapsto \beth_{n(n+1)}(\kappa)^+$ (and so the threshold for limit κ are the same as for κ^+).

Example 3.8 (χ-colored linear orders)**.** *The canonical bigness notion says that $(I, <, P_\beta)_{\beta<\chi}$ is κ-big iff $\chi \cdot \kappa \leq otp(I)$. Then $\mathcal{K}^{\chi-color}$ is a combinatorial Erdős-Rado class by Proposition 3.9.*

Proposition 3.9. *If $\lambda \xrightarrow{or} (\kappa)^n_{\mu^\chi}$, then $\lambda \xrightarrow{\chi-color} (\kappa)^n_\mu$.*

Proof: Given a coloring $c : [\chi \cdot \lambda]^n \to \mu$, we define an auxiliary coloring $d : [\lambda]^n \to {}^{\chi^n}\mu$ given by $d(\gamma_1, \dots, \gamma_n)$ is the function that maps $(i_1, \dots, i_n) \in \chi^n$ to $c(\chi\gamma_1 + i_1, \dots, \chi\gamma_n + i_n)$. There is a κ-sized homogeneous $X \subset \lambda$ by assumption. Then $\chi \cdot X$ gives a type homogeneous set.†

Example 3.10 (Trees of height $n < \omega$)**.** *The canonical bigness notion for $\mathcal{K}^{n-tr}$ is that of splitting: $I \in \mathcal{K}^{n-tr}_\mu$ iff every node of the tree on level $< n$ has $\geq \mu$-many successors. Shelah [She71] mentioned the following: for all $n, m < \omega$, there is $k(n,m) < \omega$ such that for all κ,*

$$\beth_{k(n,m)}(\kappa)^+ \xrightarrow{n-tr} (\kappa^+)^m_\kappa$$

A proof appears in [She90, Appendix, Theorem 2.6]. Alternate proofs appear in [KKS14, Theorem 6.7] and [GS86, Theorem 5.1] (in the latter, the bound on $k(n,n)$ is lowered from $2^n + n + 1$ to n^2). Thus, $\mathcal{K}^{n-tr}$ is a combinatorial Erdős-Rado class witnessed by $(\kappa^+, n) \mapsto \beth_{n^2}(\kappa)^+$.

Example 3.11 (Trees of height ω)**.** *We do not know if $\mathcal{K}^{\omega-tr}$ is a combinatorial Erdős-Rado class (although we would expect the bigness notion to be splitting). However, we are still able to show that is an Erdős-Rado class (see Corollary 5.11).*

Example 3.12 (Well-founded trees)**.** *We say that a well-founded tree is λ-big iff its cardinality is at least λ. Then [GS11, Conclusion 2.4] shows that, for every $n < \omega$ and κ*

$$(\beth_{1,n}(\kappa)) \xrightarrow{wf-tr} (\kappa)^n_\kappa$$

where $\beth_{1,n}(\lambda)$ is defined by:

- *$\beth_{1,0}(\lambda) = \lambda$ and*
- *$\beth_{1,k+1}(\lambda) = \beth_{\beth_{1,k}(\lambda)^+}(\lambda)$*

Note that the bound here is much larger than the other bounds (which are all below $\beth_\omega(\kappa)$). This impacts the witness for being an Erdős-Rado class (see Remark 4.4), but we do not know if the left-hand side here is a tight bound. Also, well-founded trees are closely related to *scattered* linear orders (those not containing a copy of $\mathbb{Q}$; see [GS11, Observation 4] building on work of Hausdorff), so this result forms a counterpoint to the nonexample coming from Sierpinski colorings.

Example 3.13 (Convexly-ordered equivalence relations)**.** *The canonical bigness notion says that $(I, <, E) \in \mathcal{K}^{ceq}$ is μ-big iff there are at least μ-many equivalence classes, each of which is of size at least μ. Proposition 3.14 below shows that $\mathcal{K}^{ceq}$ is a combinatorial Erdős-Rado class.*

Proposition 3.14. *Given infinite κ and $n < \omega$, we have*

$$\beth_{n(n+2)-1}(\kappa)^+ \xrightarrow{ceq} \left(\kappa^+\right)^n_\kappa$$

Proof: Let $(I, E, <) \in \mathcal{K}^{ceq}_{\beth_{n(n+2)-1}(\kappa)^+}$ and color it with $c : [I]^n \to \kappa$. We will use two already established facts:

$$\beth_{n(n+2)-1}(\kappa)^+ \xrightarrow{\beth_{n-1}(\kappa)^+ - or} (\beth_{n-1}(\kappa)^+)^n_\kappa$$

$$\beth_{n-1}(\kappa)^+ \xrightarrow{or} (\kappa^+)^n_\kappa$$

Then find $\{i_\alpha \in I \mid \alpha < \beth_{n-1}(\kappa)^+\}$ that are E-nonequivalent. Set $I_1 = \cup_{\alpha < \beth_{n-1}(\kappa)^+} i_\alpha / E$ and note that $(I_1, i_\alpha/E, <)_{\alpha < \beth_{n-1}(\kappa)^+} \in \mathcal{K}^{\beth_{n-1}(\kappa)^+ - or}_{\beth_{n(n+2)-1}(\kappa)^+}$. Then c still colors $[I_1]^n$, so use the result to find $I_2 \subset I_1$ and $c^* : \mathrm{S}_{\beth_{n-1}(\kappa)^+ - or} \to \kappa$ so that $(I_2, i_\alpha/E \cap I_2, <)_{\alpha < \beth_{n-1}(\kappa)^+} \in \mathcal{K}^{\beth_{n-1}(\kappa)^+ - or}_{\beth_{n-1}(\kappa)^+}$ is type-homogeneous for c with c^*.

Now consider the structure $(\{i_\alpha \mid \alpha < \beth_{n-1}(\kappa)^+\}, <) \in \mathcal{K}^{or}_{\beth_{n-1}(\kappa)^+}$. We want to give an auxiliary coloring $d : [\beth_{n-1}(\kappa)^+]^n \to {}^A\kappa$, where $A = \{s \subset {}^n(n+1) \mid \sum_{i<n} s(i) = n\}$. Then

$$d\left(\{\alpha_1 < \cdots < \alpha_n\}\right)$$

is the function that takes $s \in A$ to $c(\{j_1, \ldots, j_n\})$ for $j_1, \ldots, j_n \in I_2$ such that, for each k, $s(k)$-many of the j_ℓ's come from the equivalence class of i_{α_k}. Note that this is a well-defined coloring because I_2 was type-homogeneous for c. Then we can find $X \subset \beth_{n-1}(\kappa)^+$ of size κ^+ and $d^* : A \to \kappa$ such that X is homogeneous for d with color d^*.

Set $I_* = \{i \in I_2 \mid iEi_\alpha \text{ for some } \alpha \in X\}$, $E_* = E \restriction (I_*^2)$, and $<_* = < \restriction (I_*^2)$.

Claim: $(I_*, E_*, <_*) \in \mathcal{K}^{ceq}_{\kappa^+}$ is type-homogeneous for c. Since $|X| = \kappa^+$, I_* has κ^+-many equivalence classes. For each $\alpha \in X$, $i_\alpha / E_* = i_\alpha / E \cap I_2$ and has size at least $\beth_{n-1}(\kappa)^+ > \kappa^+$. Thus, $(I_*, E_*, <_*)$ is κ^+-big.

For homogeneity, let $j_1 <_* \cdots <_* j_n; j'_1 <_* \cdots <_* j'_n \in I_*$ have the same $\mathcal{K}^{ceq}$-type. Then these tuples are each $<$-increasing, from I_2, and are equivalent to an element of $\{i_\alpha \mid \alpha < X\}$. Because they have the same $\mathcal{K}^{ceq}$-type, there are $\alpha_1 < \cdots < \alpha_n; \alpha'_1 < \cdots < \alpha'_n$ from X that contain these witnesses and *a single map* $s \in A$ that maps ℓ to

$$|\{k \mid j_k E_* i_{\alpha_\ell}\}| = |\{k \mid j'_k E_* i_{\alpha'_\ell}\}|$$

By the homogeneity of X, we have that $d^* = d(\{\alpha_1, \ldots, \alpha_n\}) = d(\{\alpha'_1, \ldots, \alpha'_n\})$. Thus,

$$\begin{aligned} c(\{j_1, \ldots, j_n\}) &= d(\{\alpha_1, \ldots, \alpha_n\})(s) \\ &= d^*(s) \\ &= d(\{\alpha'_1, \ldots, \alpha'_n\})(s) \\ &= c(\{j'_1, \ldots, j'_n\}) \end{aligned}$$

†

Example 3.15 (n-multi-orders). *We can use $\mathcal{K}^{n-mlo}$ to point out that the choice of bigness notion is very important. If we say $(I, <_1, \ldots, <_n)$ is μ-big when $|I| \geq \mu$, then $\mathcal{K}^{n-mlo}$ is a combinatorial Erdős-Rado class simply because $\mathcal{K}^{or}$ is. However, this gives us no new information. A good bigness notion for this class should say something about the independence of the different linear orders.*

Example 3.16 (Ordered graphs). *Ordered graphs start to indicate that set theory begins to enter the picture. Hajnal and Komjáth [HK88, Theorem 12] (with correction at [HK92, Theorem 12]) show that it is consistent that there is a graph that never appears as a monochromatic subgraph. In particular, they start with a model of GCH, add a single Cohen real, and construct an uncountable bipartite graph G such that every graph H has a coloring of pairs such that there is no type-homogeneous copy of G in H. On the other hand, the next example (which subsumes this one by considering $\mathcal{K}^{(2,2)-hg}$) shows that we can consistently get a combinatorial Erdős-Rado result.*

Example 3.17 (Colored hypergraphs). *Shelah [She89, Conclusion 4.2] proved that it is consistent that an Erdős-Rado Theorem holds for the classes $\mathcal{K}^{(k,\sigma)-hg}$ with $k < \omega$. Specifically, he shows that, after an iterated forcing construction, for every $N \in \mathcal{K}^{(k,\sigma)-hg}$, $n < \omega$, and κ, there is a $M \in \mathcal{K}^{(k,\sigma)-hg}$ with $\|M\| < \beth_\omega(\|N\| + \sigma + \kappa)$ such that any coloring of $[M]^m$ with κ-many colors contains a type-homogeneous substructure isomorphic to N. For the right bigness notion, this means that*

$$\beth_\omega(\kappa) \xrightarrow{(k,\sigma)-hg} (\kappa)^n_\kappa$$

4. Erdős-Rado Classes and the Generalized Morley's Omitting Types Theorem

Erdős-Rado classes are those that allow one to build generalized indiscernibles in nonelementary classes, especially those definable in terms of type omission. Since these classes are often axiomatized in stronger logics, one could formulate the modeling property of Ramsey classes in terms of these stronger logics (in fact, Shelah [She] does this, and we compare the notions in Remark 4.7). However, this is not how order indiscernibles are typically built in AECs. Instead, we continue to work with indiscernability in a first-order (and even quantifier-free context), but strengthen the modeling property so that type omission is preserved.

Note that there are two variants of being an Erdős-Rado class here, and a few more in Definition 5.1. The cofinal variant is the most common, and gives the sharpest applications.

Definition 4.1. *Let $\mathcal{K}$ be an ordered abstract class.*

(1) *$\mathcal{K}$ is a* $(\mu, \chi, \mathbf{big})$-Erdős-Rado class *iff for every language τ of size $\leq \mu$, every $I \in \mathcal{K}^{\boldsymbol{big}}_\chi$, every τ-structure M, and every injection $f : I \to M$, there is a blueprint $\Phi \in \Upsilon^{\mathcal{K}}[\tau]$ such that*
   (a) *$\tau(\Phi) = \tau$; and*
   (b) *for each $p \in S^{inc}_{\mathcal{K}}$, there are $i_1 < \cdots < i_n \in I$ realizing p such that*
$$tp_\tau\left(f(i_1), \ldots, f(i_n); M\right) = \Phi(p)$$
(2) *$\mathcal{K}$ is a* cofinally $(\mu, \chi, \mathbf{big})$-Erdős-Rado class *iff for every language τ of size $\leq \mu$, if we have, for each cardinal $\alpha < \chi$, a τ-structure M_α, an α-*$\mathbf{big}$ *$I_\alpha \in \mathcal{K}$, and an injection $f_\alpha : I_\alpha \to M_\alpha$, then there is a blueprint $\Phi \in \Upsilon^{\mathcal{K}}[\tau]$ such that*
   (a) *$\tau(\Phi) = \tau$; and*
   (b) *for each $p \in S^{inc}_{\mathcal{K}}$, there are cofinally many $\alpha < \chi$ such that there are $i_1 < \cdots < i_n \in I_\alpha$ realizing p such that*
$$tp_\tau\left(f_\alpha(i_1), \ldots, f_\alpha(i_n); M_\alpha\right) = \Phi(p)$$

In either case, writing '$\mathcal{K}$ is [cofinally] ***big****-Erdős-Rado class' means that 'there is a function $f : Card \to Card$ such that $\mathcal{K}$ is [cofinally] $(\mu, f(\mu), \boldsymbol{big})$-Erdős-Rado for every μ.' If* $\mathbf{big}$ *is the standard bigness notion for $\mathcal{K}$, then we omit it.*

We refer to Definition 4.1.(1b) or Definition 4.1.(2b) as the *Erdős-Rado condition.* See Remark 4.7 for a comparison with Ramsey conditions.

The following is the main source of Erdős-Rado classes.

Theorem 4.2 (Generalized Morley's Omitting Types Theorem)**.** *Let $\mathcal{K}$ be combinatorially Erdős-Rado witnessed by F. Define $f : Card \to Card$ by setting $f(\mu)$ to be the first κ above*
$$\sup_{n<\omega} 2^{(\mu \cdot |S^n_{\mathcal{K}}|)}$$
such that $\alpha < \kappa$ and $n < \omega$ implies $F(\alpha, n) < \kappa$.

Then $\mathcal{K}$ is cofinally Erdős-Rado witnessed by f.

Proof: Suppose that we are given $f_\alpha : I_\alpha \to M_\alpha$ for $\alpha < f(\mu)$, where $|\tau| \leq \mu$.

We are going to build, for $n < \omega$ and $\alpha < f(\mu)$

- $\Phi_n : \mathrm{S}^{inc,n}_{\mathcal{K}} \to S^n_\tau$;
- $\beta_n(\alpha) < f(\mu)$;
- $\gamma_{n+1}(\alpha) < f(\mu)$;
- α-$\mathbf{big}$ $I^n_\alpha \in \mathcal{K}$;
- $h^{n+1}_\alpha : I^{n+1}_\alpha \to I^n_{\gamma_{n+1}(\alpha)}$
- $f^n_\alpha : I^n_\alpha \to M_{\beta_n(\alpha)}$

such that

(1) for each $\alpha < f(\mu)$ and $i_1 < \cdots < i_n \in I^n_\alpha$, we have that
$$\Phi_n\left(tp_{\mathcal{K}}(i_1, \ldots, i_n; I^n_\alpha)\right) = tp_\tau\left(f^n_\alpha(i_1), \ldots, f^n_\alpha(i_n); M_{\beta_n(\alpha)}\right)$$

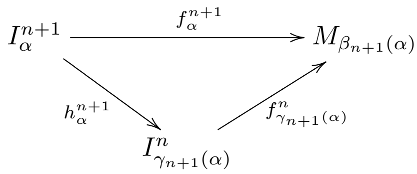
(2) the Φ_n are coherent in the following sense: if $p \in S_{\mathcal{K}}^{inc,n}$ and $s \subset n$, then

$$\Phi_n(p)^s = \Phi_{|s|}(p^s)$$

(see Definition 2.2.(5) for this notation)

(3) for every $\alpha < f(\mu)$ and $n < \omega$, $\alpha \leq \beta_n(\alpha)$ and $\alpha \leq \gamma_n(\alpha)$; and

(4) for every $\alpha < f(\mu)$ and $n < \omega$, we have that $\beta_{n+1}(\alpha) = \beta_n(\gamma_{n+1}(\alpha))$ and the following commutes



This is enough: Set $\Phi := \cup_{n<\omega}\Phi_n$. Then this is a function with domain $S_{\mathcal{K}}$ and range S_τ. Moreover, the coherence condition implies that it is proper for $\mathcal{K}$. Now we wish to show that it has the type reflection required by the Erdős-Rado condition, see Defintion 4.1.(1b).

Let $p \in S_{\mathcal{K}}^{inc,n}$ and $\alpha_0 < f(\mu)$. $I^n_{\alpha_0+1}$ is α-**big**, so there is $i_1 < \cdots < i_n \in I^n_{\alpha_0+1}$ realizing p. Then, by (1) of the construction

$$\Phi(p) = \Phi_n\left(tp_{\mathcal{K}}(i_1, \ldots, i_n; I^n_{\alpha_0+1})\right) = tp_\tau\left(f^n_{\alpha_0+1}(i_1), \ldots, f^n_{\alpha_0+1}(i_n); M_{\beta_n(\alpha_0+1)}\right)$$

If $n = 0$, then $I^0_{\alpha_0+1} = I^0$ and we are done. If $n > 0$, then composing the h-embeddings, we get $h^* : I^n_{\alpha_0+1} \to I_{\beta_n(\alpha_0+1)}$ such that $f^n_\alpha = f_{\beta_n(\alpha_0+1)} \circ h^*$. Thus, $h^*(i_1), \ldots, h^*(i_n) \in I_{\beta_n(\alpha_0+1)}$ realize p and

$$\Phi(p) = tp_\tau\left(f_{\beta_n(\alpha_0+1)}\left(h^*(i_1)\right), \ldots, f_{\beta_n(\alpha_0+1)}\left(h^*(i_n)\right); M_{\beta_n(\alpha_0+1)}\right)$$

Since $\beta_n(\alpha_0+1) > \alpha_0$, this completes the proof.

Construction: We work by induction on n. For $n = 0$, we use what we are given: $\Phi_0 = \emptyset$; $\beta_0(\alpha) = \alpha$; $I^0_\alpha = I_\alpha$; and $f^0_\alpha = f_\alpha$.

For $n+1$, suppose we have completed the construction up to stage n. Fix $\alpha < f(\mu)$. WLOG $\alpha > 2^\mu \geq |S^{n+1}_\tau|$; otherwise, use replace α with $\alpha + (2^\mu)^+$. Then $F(\alpha, n+1) < f(\mu)$. Consider the coloring

$$c^{n+1}_\alpha : [I^n_{F(\alpha,n+1)}]^{n+1} \to S^{n+1}_\tau$$

given by

$$c^{n+1}_\alpha\left(\{i_1 < \cdots < i_{n+1}\}\right) = tp_\tau\left(f^n_{F(\alpha,n+1)}(i_1), \ldots, f^n_{F(\alpha,n+1)}(i_n); M_{\beta_n(F(\alpha,n+1))}\right)$$

where $\{i_1 < \cdots < i_{n+1}\}$ indicates that the unordered set $\{i_1, \ldots, i_{n+1}\}$ is indexed so it occurs in increasing order as $i_1 < \cdots < i_{n+1}$. Recall that $F(\alpha, n+1) \to^{\mathcal{K}} (\alpha)^{n+1}_{2^\mu}$. Use this to find $\bar{I}^{n+1}_\alpha \in \mathcal{K}$ that is α-big; $\bar{h}^{n+1}_\alpha : \bar{I}^{n+1}_\alpha \to I^n_{F(\alpha,n+1)}$; and $c^{*,n+1}_\alpha : S^{inc,n+1}_{\mathcal{K}} \to S^{n+1}_\tau$ such that, for all $i_1 < \cdots < i_n \in \bar{I}^{n+1}_\alpha$, we have

$$tp^\tau_{qf}\left(f^n_{F(\alpha,n+1)} \circ \bar{h}^{n+1}_\alpha(i_1), \ldots, f^{n+1}_{F(\alpha,n)} \circ \bar{h}^{n+1}_\alpha(i_{n+1}); M_{\beta_n(F(\alpha,n+1))}\right) = c^{*,n+1}_\alpha\left(tp^{\mathcal{K}}_{qf}(i_1, \ldots, i_{n+1}t; \bar{I}^{n+1}_\alpha)\right)$$

For each α, we have built $c^{*,n+1}_\alpha \in {}^{S^{inc,n+1}_{\mathcal{K}}}S^{n+1}_\tau$. Since $f(\mu)$ is greater than the number of such functions, there is $X \subset f(\mu)$ of size $f(\mu)$ and $c^{*,n+1} : S^{inc,n+1}_{\mathcal{K}} \to S^{inc,n+1}_\tau$ such that, for all $\alpha \in X$, $c^{*,n+1}_\alpha = c^{*,n+1}$. Set $\pi_{n+1} : X \cong f(\mu)$ to be the collapse of X onto its order type; note that $\alpha \leq \pi^{-1}_{n+1}(\alpha)$ for all $\alpha \in X$.

Set

(1) $\Phi_{n+1} = c^{*,n+1}$

(2) $I^{n+1}_\alpha = \bar{I}^{n+1}_{\pi^{-1}(\alpha)}$

(3) $\gamma_{n+1}(\alpha) = F\left(\pi^{-1}(\alpha), n+1\right)$
(4) $\beta_{n+1}(\alpha) = \beta_n\left(F(\pi^{-1}(\alpha), n+1)\right)$
(5) $h^{n+1}_\alpha = \bar{h}_{\pi^{-1}(\alpha), n+1}$
(6) $f^{n+1}_\alpha = f^n_{F(\pi^{-1}(\alpha), n+1)} \circ h^{n+1}_\alpha$

These satisfy the pieces of the construction: by induction, each of the $c^{*,n+1}_\alpha$'s extend $c^{*,n}$ in the sense that the restriction to n-types is determined by $c^{*,n}$. This gives the coherence. The other properties are routine to verify. †

Corollary 4.3. *Each of the examples of combinatorial Erdős-Rado classes in Section 3.1 are cofinal Erdős-Rado classes.*

Remark 4.4. *Whenever $F(\mu, n) \leq \beth_\omega(\mu)$, then this gives the bound $f(\mu) = \beth_{(2^\mu)^+}$ that often appears in the theory of nonelementary classes. In the case of well-founded trees, we get the bound $\beth_{1,(2^\mu)^+}$. These bounds can be improved by phrasing in terms of the undefinability of well-ordering of certain PC classes (this is done for specific cases in [She90, GS86]).*

The following extends the normal notion of PC classes to include classes with a strong substructure relation. Note that Chang's Presentation Theorem [Cha68] implies any $\mathbb{L}_{\infty,\omega}$-axiomatizable class with 'elementary according to a fragment' as the strong substructure is what we will call a *PC pair*, and Shelah's Presentation Theorem [She87a] extends this to Abstract Elementary Classes.

Definition 4.5.

(1) *Let $\mathbb{K}$ be an abstract class with $\tau = \tau(\mathbb{K})$. $\mathbb{K}$ is a* PC class *iff there is a language $\tau_1 \supset \tau$, a (first-order) τ_1-theory T_1, and a collection Γ of τ_1-types such that, for any τ-structure M, $M \in \mathbb{K}$ iff there is an expansion M_1 of M to τ_1 that models T_1 and omits all types in Γ.*
(2) *Let $\mathbb{K}$ be a class of τ-structures and $\prec_\mathbb{K}$ be a partial order on $\mathbb{K}$. $(\mathbb{K}, \prec_\mathbb{K})$ is a* PC pair *iff there is a language $\tau_1 \supset \tau$, a τ_1-theory T_1, and a collection Γ of τ_1-types such that*
   - *for any τ-structure M, $M \in \mathbb{K}$ iff there is an expansion M_1 of M to τ_1 that models T_1 and omits all types in Γ; and*
   - *for any $M, N \in \mathbb{K}$, $M \prec N$ iff there are expansions M_1 of M and N_1 of N to τ_1 that models T_1 and omits all types in Γ such that $M_1 \subset N_1$*

Theorem 4.6. *Let $\mathcal{K}$ be a cofinally Erdős-Rado class witnessed by f and let $\mathbb{K}$ be a PC pair with $\tau = \tau(\mathbb{K})$ and τ_1 the witnessing language. Suppose that, for every $\alpha < f(|\tau_1|)$, there is $M_\alpha \in \mathbb{K}$; α-big $I_\alpha \in \mathcal{K}$; and $f_\alpha : I_\alpha \to M_\alpha$. Then, there is $\Phi \in \Upsilon^{\mathcal{K}}_{|\tau_1|}[\mathbb{K}]$ such that, for every $p \in S^{inc}_{\mathcal{K}}$, there are cofinally many $\alpha < f(|\tau_1|)$ such that there are $i_1 < \cdots < i_n \in I_\alpha$ realizing p such that*

$$tp_\tau\left(f_\alpha(i_1), \ldots, f_\alpha(i_n)/\emptyset; M_\alpha\right) = \Phi(p)$$

Moreover, Φ also determines Galois types in the following sense: if $\sigma_1, \ldots, \sigma_k$ are $\tau(\Phi)$-terms; $I, J \in \mathcal{K}$; and $i_1, \ldots, i_n \in I$ and $j_1, \ldots, j_n \in J$ are tuples such that

$$tp_\mathcal{K}\left(i_1, \ldots, i_n; I\right) = tp_\mathcal{K}\left(j_1, \ldots, j_n; J\right)$$

then

$$\begin{aligned} & tp_\mathbb{K}\left(\sigma_1(i_1, \ldots, i_n), \ldots, \sigma_k(i_1, \ldots, i_n); EM_{\tau(\mathbb{K})}(I, \Phi)\right) \\ = {} & tp_\mathbb{K}\left(\sigma_1(j_1, \ldots, j_n), \ldots, \sigma_k(i_1, \ldots, i_n); EM_{\tau(\mathbb{K})}(J, \Phi)\right) \end{aligned}$$

In particular, $I \subset EM_\tau(I, \Phi)$ is a collection of $\mathcal{K}$-indiscernibles.

A version of Theorem 4.6 also holds for Erdős-Rado classes (without the cofinal adjective) when there is a single embedding from a $f(|\tau_1|)$-big member of $\mathcal{K}$ into M.

Proof: Let T_1 and Γ in the language τ_1 witness that $\mathbb{K}$ is a PC pair. Let $f_\alpha : I_\alpha \to M_\alpha$ for $\alpha < f(|\tau_1|)$ as in the hypothesis. By a further Skolem expansion, we can assume that T_1 and Γ are universal. Then we can expand M_α to M^*_α, which is a τ_1-structure satisfying T_1 and omitting Γ. Since $\mathcal{K}$ is $(|\tau_1|, f(|\tau_1|))$-cofinally Erdős-Rado, we can find a blueprint $\Phi \in \Upsilon^{\mathcal{K}}[\tau_1]$ satisfying the Erdős-Rado condition, Defintion 4.1.(1b).

First, we wish to show that Φ is proper for $(\mathcal{K}, \mathbb{K})$. For membership in $\mathbb{K}$, let $I \in \mathcal{K}$. It suffices to show that any universal formula that fails of a tuple in $EM(I, \Phi)$ is already false of some tuple in some M_α. So suppose that $\phi(\mathbf{x}, \mathbf{y})$ is quantifier free and $\mathbf{a} \in EM(I, \Phi)$ are such that $EM(I, \Phi) \vDash \neg\forall \mathbf{y} \phi(\mathbf{a}, \mathbf{y})$. Then there is $\mathbf{b} \in EM(I, \Phi)$ such that $EM(I, \Phi) \vDash \neg\phi(\mathbf{a}, \mathbf{b})$. Since $EM(I, \Phi)$ is generated by τ_1-terms, there are τ_1-terms $\sigma_1, \ldots, \sigma_n$ and $i_1, \ldots, i_k \in I$ such that $\mathbf{a}, \mathbf{b} = \sigma_1^{EM(I,\Phi)}(\mathbf{i}), \ldots, \sigma_n^{EM(I,\Phi)}(\mathbf{i})$; without loss, ther terms are such that $i_1 < \cdots < i_k$.

Set $p = tp_{\mathcal{K}}(\mathbf{i}; I)$. By the Erdős-Rado condition, there is some $\alpha < f(\mu)$ and $j_1 < \cdots < j_k$ such that

$$tp_{\tau_1}(i_1, \ldots, i_k; EM(I, \Phi)) = \Phi(p) = tp_{\tau_1}(f_\alpha(j_1), \ldots, f_\alpha(j_k); M^*_\alpha)$$

In particular,

$$M^*_\alpha \vDash \neg\phi(\sigma_1(\mathbf{j}), \ldots, \sigma_n(\mathbf{j}))$$

But this contradicts that M^*_α models T_1 and omits Γ.

For substructure, this follows from the definition for PC pair and the fact that Φ is proper for $(\mathcal{K}, \mathbb{K}^{\tau_1})$.

For the moreover, applying the $EM_\tau(\cdot, \Phi)$ map to the diagram witnessing type equality in $\mathcal{K}$ witnesses the type equality in $\mathbb{K}$. †

†

Remark 4.7. *We want to highlight the differences between the Erdős-Rado condition (Definition 4.1.(1b)) to the relevant condition in uses of Ramsey classes, such as [She, Definition 1.15] or [GHS, Definition 2.12]. Rephrased to our language to highlight the comparison, the Ramsey modeling condition is*

given $p \in S^{\mathcal{K}_0}_{qf}$*, if* $\phi(x_1, \ldots, x_n)$ *is a quantifier-free formula*[3] *in* τ_1 *such that* for all $i_1, \ldots, i_n \in I$ *that satisfy p, we have*

$$M \vDash \phi(f(i_1), \ldots, f(i_n))$$

then we have

$$\phi(x_1, \ldots, x_n) \in \Phi(p)$$

The key difference is the following:

- *Ramsey classes build blueprints that are only required to reflect the structure that occurs everywhere*
- *Erdős-Rado classes build blueprints that fully reflect the structure happening somewhere*

This makes Ramsey classes ill-equipped to handle type omission and nonelementary classes. These is because, after Skolemization, the generating sequence might not agree on where terms omit the types, so the blueprint is not guaranteed to omit types. Shelah [She, Definition 1.15] addresses this by introducing $\mathcal{L}$-nice Ramsey classes (for a logic fragment $\mathcal{L}$) that considers formulas in $\mathcal{L}$. However, it is unclear how to get a $\mathcal{L}$-nice Ramsey class outside of Erdős-Rado classes. He also considers the notion of a strongly Ramsey class, which is similar to our notion.

[3]As always, this means 'quantifier-free in $\mathbb{L}_{\omega,\omega}$.'

5. Further results

5.1. **Reversing Generalized Morley's Omitting Types Theorem.** We would like to have a converse to the Generalized Morley's Omitting Types Theorem 4.2 that says that all Erdős-Rado classes come from a combinatorial result. However, this seems unlikely to be true (and we discuss candidates for this in Section 5.4). The issue is that the definition of a (cofinally) Erdős-Rado class is not as tied to the relevant bigness notion as the definition of combinatorial Erdős-Rado classes. In particular, the definition leaves open the possibility that there is only a single witness to the Erdős-Rado condition, while combinatorial Erdős-Rado classes require a big set of witnesses to the type-homogeneity. If we strengthen this requirement, then we get a converse.

Definition 5.1. *We say that $\mathcal{K}$ is* strongly $(\mu, \chi, \mathbf{big})$-Erdős-Rado *iff for all $I \in \mathcal{K}^{\mathbf{big}}_\chi$ and every $f : I \to M$ with $|\tau(M)| \leq \mu$, there is a blueprint $\Phi \in \Upsilon^{\mathcal{K}}[\tau]$ such that $\tau(\Phi) = \tau(M)$ such that for all $\alpha < \chi$ and $n < \omega$, there is an α-big $I^n_\alpha \leq_{\mathcal{K}} I$ such that, for every $i_1 < \cdots < i_n \in I^n_\alpha$, we have*

$$tp_{\tau(M)}\left(f(i_1), \ldots, f(i_n); M\right) = \Phi\left(tp_{\mathcal{K}}(i_1, \ldots, i_n; I)\right)$$

We define the cofinal variant and what it means to omit the $(\mu, \chi, \mathbf{big})$-prefix as in Definition 4.1.

Theorem 5.2. *Let $\mathcal{K}$ be an ordered abstract class.*

(1) *If $\mathcal{K}$ is combinatorially Erdős-Rado witnessed by F, then $\mathcal{K}$ is strongly, cofinally Erdős-Rado witnessed by $\mu \mapsto \sup_{n<\omega} F(\mu, n)$.*
(2) *If $\mathcal{K}$ is strongly, cofinally $(\mu, \chi, \mathbf{big})$-Erdős-Rado, then $\mathcal{K}$ is strongly $(\mu, \chi, \mathbf{big})$-Erdős-Rado.*
(3) *If $\mathcal{K}$ is strongly $(\mu, \chi, \mathbf{big})$-Erdős-Rado, then, for each $n < \omega$ and $\lambda < \chi$,*

$$(\chi) \xrightarrow[\mathbf{big}]{\mathcal{K}} (\lambda)^n_\mu$$

(4) *If $\mathcal{K}$ is strongly Erdős-Rado witness by f, then $\mathcal{K}$ is combinatorially Erdős-Rado witnessed by $F(\mu^+, n) = f(\mu)$.*

Proof: The proof of the Generalized Morley's Omitting Types Theorem 4.2 proves (1): the I^n_α built in that proof are exactly the ones needed to witness 'strong.' The proof of (2) is straightforward. We prove (3), which is enough to prove (4). The idea is that a potential coloring is turned into a structure, and the derived blueprint is used to figure out the colors for the large set.

Let $\lambda < \chi$ and $c : [I]^n \to \mu$ be a coloring of $I \in \mathcal{K}^{\mathbf{big}}_\chi$. We build this into a two-sorted structure

$$M = \langle I, \mu; c, \alpha \rangle_{\alpha < \kappa}$$

We have an embedding $f : I \to M$ given by the identity. Then $|\tau(M)| = \mu$, so the strong Erdős-Rado property gives us a blueprint $\Phi : \mathrm{S}^{inc}_{\mathcal{K}} \to \mathrm{S}_{\tau(M)}$ as in Definition 5.1.

Claim 1: For every $p \in \mathrm{S}^{inc,n}_{\mathcal{K}}$, there is a unique $\alpha_p < \mu$ such that "$c(x_1, \ldots, x_n) = \alpha_p$" $\in \Phi(p)$.

Take $I^\omega_n \leq_{\mathcal{K}} I$ witnessing the strong Erdős-Rado property and find $i_1 < \cdots < i_n \in I^\omega_n$ realizing p; such a tuple exists by the definition of a bigness notion. Then $\Phi(p) = tp_{\tau(M)}(i_1, \ldots, i_n; M)$. This has a color, so $\alpha_p = c(i_1, \ldots, i_n)$. $\dagger_{\text{Claim 1}}$

Set $c^* : \mathrm{S}^{inc,n}_{\mathcal{K}} \to \mu$ to be the function that takes p to α_p.

Claim 2: I^n_λ is type-homogeneous for c as witnessed by c^*.

Straightforward. $\dagger_{\text{Claim 2}}$

Since I^n_λ is λ-**big**, this proves the theorem. $\dagger_{\text{Theorem 5.2}}$

Note that this is not an exact converse because there is some slippage in the witnessing functions: starting with a combinatorial Erdős-Rado class witnessed by F and applying the above theorem gives a combinatorial Erdős-Rado class witnessed by $(\mu, n) \mapsto \sup_{k<\omega} F(\mu, k)$. However, this doesn't affect the bounds on the Erdős-Rado class.

5.2. **A category theoretic interpretation of blueprints.** This section gives a category theoretic perspective on the results we've proven and indiscernibles in general. It requires more category theoretic background than the rest of the paper (such as [MP89] or [AR94]), but can be skipped.

Makkai and Paré give the following statement credited to Morley.

Fact 5.3 ([MP89, Theorem 3.4.1])**.** *$\mathcal{K}^{or}$ is a "minimal" large, $\mathbb{L}_{\infty,\omega}$-elementary category. This means that if $\mathbb{K}$ is a large, $\mathbb{L}_{\infty,\omega}$-elementary category, then there is a faithful functor $\Phi : \mathcal{K}^{or} \to \mathbb{K}$ that preserves directed colimits.*

This is not phrased as Morley (likely) ever wrote it, but this is the classic proof of Morley's Omitting Types Theorem. The functor Φ is simply the blueprint that takes $I \in \mathcal{K}^{or}$ to $EM_\tau(I, \Phi) \in \mathbb{K}$. With generalized indiscernibles in hand, we have a generalization.

Theorem 5.4. *Erdős-Rado classes are minimal amongst the large, $\mathbb{L}_{\infty,\omega}$-elementary categories (in the sense of Fact 5.3).*

We include a proof to make the translation more clear (and in part because Makkai and Paré do not give a proof).

Proof: Let $\mathcal{K}$ be an Erdős-Rado class and $\mathbb{K}$ be a large, $\mathbb{L}_{\infty,\omega}$-elementary category. This means that there is a theory $T \subset \mathbb{L}_{\infty,\omega}(\tau)$ such that $\mathbb{K}$ is (equivalent to) Mod T. Fix $f : Card \to Card$ witnessing that $\mathcal{K}$ is Erdős-Rado. By virtue of being large, there is $M \in \mathbb{K}$ such that $\|M\| \geq f(\mu)$. Thus by Theorem 4.6 and Chang's Presentation Theorem, there is a blueprint $\Phi \in \Upsilon^{\mathcal{K}}[\mathbb{K}]$. Define a functor $F : \mathcal{K} \to \mathbb{K}$ by, for $I \in \mathcal{K}$, $F(I) = EM_\tau(I, \Phi)$ and, for $f : I \to J$ in $\mathcal{K}$, Ff the map that takes $\sigma^{EM(I,\Phi)}(i_1, \dots, i_n)$ for a $\tau(\Phi)$-term σ to $\sigma^{EM(J,\Phi)}(f(i_1), \dots, f(i_n))$.

This is clearly faithful. Moreover, the EM construction commutes with colimits, so F preserves them. $\dagger$

This proof works by noting that blueprints can be seen as well-behaved functors. We can actually specify the properties of these functors to obtain a converse. The one additional property that we need is that the size of $EM_\tau(I, \Phi)$ is determined by $|I|$ and an additional cardinal parameter representing $|\tau(\Phi)|$. The following is based on an argument developed with John Baldwin in the case $\mathcal{K} = \mathcal{K}^{or}$.

Theorem 5.5. *Suppose $\mathcal{K}$ is a universal Erdős-Rado class and $\mathbb{K}$ is a large, $\mathbb{L}_{\infty,\omega}$-elementary category. Let $F : \mathcal{K} \to \mathbb{K}$ be a faithful functor that preserves directed colimits such that there is a cardinal μ_F so that $\|F(I)\| = |I| + \mu_F$ for every $I \in \mathcal{K}$. Then there is a blueprint $\Phi \in \Upsilon^{\mathcal{K}}_{\mu_F}[\mathbb{K}]$ such that the functor induced by $I \in \mathcal{K} \mapsto EM_\tau(I, \Phi)$ is naturally isomorphic to F.*

Proof: Let $T \subset \mathbb{L}_{\infty,\omega}(\tau)$ such that $\mathbb{K}$ is (equivalent to) Mod T. Enumerate the $\mathcal{K}$-types as $\langle p^n_i \in \mathrm{S}^n_{\mathcal{K}} \mid i < \mu_n \rangle$, and pick some $I^n_i \in \mathcal{K}$ that is generated by elements $a^{i,n}_1, \dots, a^{i,n}_n$ that realize p^n_i. We expand each $F(I^n_i)$ to a $\tau^* := \tau(\mathbb{K}) \cup \{F^n_\alpha : \alpha < \mu_F, n < \omega\}$-structure as in Shelah's Presentation Theorem. In fact, we only give an explicit description of the $\{F^n_\alpha : \alpha < \mu_F\}$ structure on the $F(I^n_i)$: for each $n < \omega$ and $i < \mu_n$, define these functions so that

$\{F^n_\alpha(a^{i,n}_1, \ldots, a^{i,n}_n) : \alpha < \mu\}$ enumerates the universe of $F(I^n_i)$. Then define the remaining functions arbitrarily.

Since F preserves directed colimits and $\mathcal{K}$ is generated by the I^n_i under directed colimits, we can lift these expansions to the rest of $F"\mathcal{K}$. Taking I large enough, we can define a blueprint $\Phi \in \Upsilon^{\mathcal{K}}[\tau^*]$. While $EM(I, \Phi)$ and $F(I)$ likely differ, this difference only occurs in the expanded langauge $\tau^* - \tau$. Thus, Φ is as desired. †

Note that this converse requires that models be of a predictable size. Specializing to linear order, we demand that $\Phi(n)$ be the same for all $n < \omega$. This is necessary for the formalism we've given where $\tau(\Phi)$ consists of functions that can be applied to any element of $EM(I, \Phi)$. To state the most general result, we could change this to only apply the functions of $\tau(\Phi)$ to the skeleton I. Then the different sizes of $\Phi(n)$ could be dealt with by having different numbers of functions of different cardinalities. But this seems like a marginal gain after what would be significant technical pain. Additionally, the requirement that $\mathcal{K}$ be universal can be removed.

We return to this category theoretic perspective in Section 6.2 when discussing indiscernible collapse.

5.3. **Generalized Shelah's Omitting Types Theorem.** The application of Morley's Omitting Types Theorem to Abstract Elementary Classes is normally done through Shelah's Presentation Theorem, which gives a type omitting characterization of these classes (see Theorem 4.6 for this argument). Moving beyond this, Shelah has proved an omitting types theorem that strengthens this and specifically applies to to Abstract Elementary Classes in that it references Galois types rather than syntactic types ([MS90] and [She99, Lemma 8.7] both use some version of this in the same year). The key addition is a reduction in the cardinal threshold for type omission at the cost of less control over what types are omitted. The main combinatorial tool is still using the Erdős-Rado Theorem to build Ehrenfeucht-Mostowski models, so we can similarly prove a version for any Erdős-Rado class.

One nonstandard piece of notation is necessary.

Definition 5.6. *Suppose $\mathbb{K}$ is an AEC, and let $N \prec_{\mathbb{K}} M$, $p \in S_{\mathbb{K}}(N)$, and $\chi \leq \|N\|$. We say M* omits p/E_χ *iff, for every $c \in M$, there is some $N_0 \prec_{\mathbb{K}} N$ of size $< \chi$ such that c does not realize $p \restriction N_0$.*

Theorem 5.7 (Generalized Shelah's Omitting Types Theorem)**.** *Let $\mathcal{K}$ be an Erdős-Rado class and $\mathbb{K}$ be an Abstract Elementary Class and $|\tau(\mathcal{K})| + LS(\mathbb{K}) \leq \chi \leq \lambda$ with*

1. *$f(\mu) < \beth_{LS(\mathbb{K})}(\mu)$ where f witnesses that $\mathcal{K}$ is Erdős-Rado (for simplicity);*
2. *$N_0 \prec_{\mathbb{K}} N_1$ with $\|N_0\| \leq \chi$ and $\|N_1\| = \lambda$;*
3. *$\Gamma_0 = \{p^0_i : i < i^*_0\} \subset S_{\mathbb{K}}(N_0)$; and*
4. *$\Gamma_1 = \{p^1_i : i < i^*_1\} \subset S_{\mathbb{K}}(N_1)$ with $i^*_1 \leq \chi$.*

Suppose that, for each $\alpha < (2^\chi)^+$, there is $M_\alpha \in \mathbb{K}$ such that

1. *$f^0_\alpha : I^0_\alpha \to M_\alpha$ for $I^0_\alpha \in \mathcal{K}_{\beth_\alpha(\lambda)}$;*
2. *$N_1 \prec M_\alpha$;*
3. *M_α omits Γ_0; and*
4. *M_α omits p^1_i/E_χ for each $i < i^*_1$.*

Then there is $\Phi \in \Upsilon^{\mathcal{K}}[\mathbb{K}]$; increasing, continuous $\{N'_q \in \mathbb{K}_{\leq \chi} \mid q \in S^{inc}_{\mathcal{K}}\}$; and increasing Galois types $p^1_{i,q} \in S_{\mathbb{K}}(N'_q)$ for $q \in S^{inc}_{\mathcal{K}}$ and $i < i^*_1$ such that

1. *$N_0 = N'_0 = EM_\tau(\emptyset, \Phi)$;*
2. *for each $q \in S^{inc}_{\mathcal{K}}$, there is $\mathbf{i}^q \in I \in \mathcal{K}$ realizing q such that $N'_q \prec_{\mathbb{K}} EM_\tau(\mathbf{i}^q, \Phi)$ and $f_q : EM_\tau(\mathbf{i}^q, \Phi) \to M_{\alpha_q}$ for some $\alpha_q < (2^\chi)^+$ such that $f_q(N'_q) \prec_{\mathbb{K}} N_1$;*

(3) $p^1_{i,q} = f_q^{-1}\left(p^1_i \restriction f_q(N'_q)\right)$; *and*
(4) *For every* $I \in \mathcal{K}_\omega$, $EM_\tau(I, \Phi)$ *omits every type in* Γ_0 *and omits any type that extends* $\{p^1_{i,q} : q \in S^{inc}_{\mathcal{K}}\}$ *in the following strong sense: if* $a \in EM_\tau(I, \Phi)$ *is in the* $\tau(\Phi)$*-closure of* $\mathbf{i} \in I$*, then a doesn't realize* $H(p^1)i, q)$*, where* $H : EM_\tau(\mathbf{i}^q, \Phi) \cong EM_\tau(\mathbf{i}, \Phi)$ *is the lifting of* $\mathbf{i} \mapsto \mathbf{i}^q$.

The proof of the above adapts the proof of Shelah's Omitting Types Theorem just as Theorem 4.2 adapts Morley's original; see the notes by the author for a very detailed proof of (the ordinary) Shelah's Omitting Types Theorem [Bon].

5.4. **End-approximations.** Although, most classes are proved to be cofinally Erdős-Rado by proving a combinatorial theorem and then applying Generalized Morley's Omitting Types Theorem 4.2, the example of $\mathcal{K}^{\omega-tr}$ gives a case where this doesn't happen. Instead, the fact that they are approximated by the trees of height n, each of which satisfy a combinatorial theorem, allows us to show they are a cofinal Erdős-Rado class. We give an abstract condition in Defintion 5.8 below, and Proposition 5.9 shows that $\mathcal{K}^{\omega-tr}$ fits into this framework.

Definition 5.8. *We say that* $\mathcal{K}$ *is* end-approximated by combinatorial Erdős-Rado classes *iff there are combinatorial Erdős-Rado classes* $\{\mathcal{K}^n \mid n < \omega\}$ *such that*
(1) $\tau(\mathcal{K}^n) \subset \tau(\mathcal{K}^{n+1})$ *and* $\tau(\mathcal{K}) = \cup_{n<\omega}\tau(\mathcal{K}^n)$;
(2) *there are coherent restriction maps*
$$\cdot \restriction n : \mathcal{K}^{\geq n} \to \mathcal{K}^n$$
where these restriction maps might cut out the universe as well as the language;
(3) $S_{\mathcal{K}}$ *is the limit of the* $S^{\leq n}_{\mathcal{K}^n}$ *for* $n < \omega$ *(this means that the* $p \in S_{\mathcal{K}}$ *can be identified with sequences* $\langle p^n \in S_{\mathcal{K}^n} \mid k_p \leq n < \omega\rangle$ *such that* $k_p \leq m < n$ *implies* $p^n \restriction m = p^m$*;*
(4) *the restriction of an* α*-big model (with bigness computed in the domain) is* α*-big (in the restricted class); and*
(5) *if* $I \in \mathcal{K}^n_\alpha, J \in K_\alpha$*, and* $f : I \to J \restriction n$ *is a* $\mathcal{K}^n$*-embedding, then this can be lifted to* $\hat{I} \in \mathcal{K}^{n+1}_\alpha$ *and* $\hat{f} : \hat{I} \to J \restriction (n+1)$ *such that* $\hat{I} \restriction n = I$ *and* $\hat{f} \restriction I = f$.

The lifting condition Definition 5.8.(5) is the key property. Our initial motivation for this framework was to show that $\mathcal{K}^{\omega-tr}$ is an Erdős-Rado class. After sending him a draft, Baldwin pointed us to [BS12, Section 4], where this is already shown. However, this framework can also be used to show that $\mathcal{K}^{(\omega,\sigma)-hg}$ is an Erdős-Rado class in Shelah's model in [She89, Conclusion 4.2] (specifically, from the conclusion that $\mathcal{K}^{(k,\sigma)-hg}$ is an Erdős-Rado class for each $k < \omega$).

Proposition 5.9. $\mathcal{K}^{\omega-tr}$ *is end approximated by* $\{\mathcal{K}^{n-tr} \mid n < \omega\}$.

Proof: The proof is straightforward. The truncation map $\cdot \restriction n$ truncates a tree of height $\geq n$ to its $\leq n$ levels. Any $p \in S^n_{\mathcal{K}^{\omega-tr}}$ specifies the max height k_p of a realization.

For condition (5), let I, J, f be as there. Build $\hat{I}$ by specifying $\hat{I} \restriction n = I$ and, given maximal $\eta \in I$, the successors of η in $\hat{I}$ are an isomorphic copy the successors of $f(\eta)$ in J. Since J is at least α-splitting, so is $\hat{I} \in \mathcal{K}^{n+1}_\alpha$, and the isomorphisms give the lift $\hat{f} : \hat{I} \to J \restriction n+1$. †

Theorem 5.10. *Let* $\mathcal{K}$ *be end-approximated by combinatorial Erdős-Rado classes. Then* $\mathcal{K}$ *is a cofinal Erdős-Rado class. A witnessing function* f *is the one taking* μ *to the first cardinal above*
$$\sup_{n<\omega} 2^{(\mu \cdot |S^n_{\mathcal{K}}|)}$$
that is closed under the application of the functions witnessing that each $\mathcal{K}^n$ *is combinatorial Erdős-Rado.*

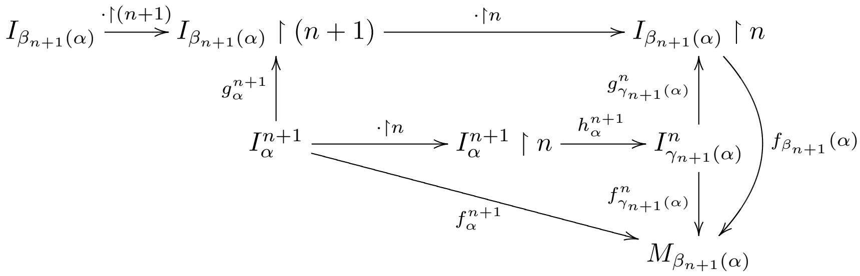
Proof: We follow the strategy of the proof of Generalized Morley's Omitting Types Theorem 4.2 and highlight the differences.

We want to build

- $\Phi_n : \mathrm{S}^n_{\mathcal{K}^n} \to \mathrm{S}^n_\tau$
- $\beta_n(\alpha), \gamma_{n+1}(\alpha) < f(\mu)$
- $I^n_\alpha \in \mathcal{K}^n_\alpha$
- $h^{n+1}_\alpha : I^{n+1}_\alpha \upharpoonright n \to I^n_{\gamma_{n+1}(\alpha)}$, a $\mathcal{K}^n$-embedding
- $g^n_\alpha : I^n_\alpha \to I_{\beta_n(\alpha)} \upharpoonright n$
- $f^n_\alpha : I^n_\alpha \to M_{\beta_n(\alpha)}$

such that all of the same conditions hold except that (4) is replaced by

(∗) for every $\alpha < f(\mu)$ and $n < \omega$, we have that $\beta_n(\gamma_{n+1}(\alpha)) = \beta_{n+1}(\alpha)$ and the following commutes



<ins>$n = 0$</ins>: Easy, just like before.

<ins>$n + 1$</ins>: As before, suppose we have for n. For each $\alpha < f(\mu)$, consider the $\mathcal{K}^n$-embedding $g^n_\alpha : I^n_\alpha \to I_{\beta_n(\alpha)} \upharpoonright n$. Using item (5) of end-approximation, we can lift to $\hat{g}^{n+1}_\alpha : \hat{I}^{n+1}_\alpha \to I_{\beta_n(\alpha)} \upharpoonright (n+1)$ with $\hat{I}^{n+1}_\alpha \in \mathcal{K}^{n+1}_\alpha$ and $I^{n+1}_\alpha \upharpoonright n = I^n_\alpha$.

Color $c^{n+1}_\alpha : [\hat{I}^{n+1}_{F_{n+1}(\alpha,n+1)}]^{n+1} \to \mathrm{S}^{n+1}_\tau$ by

$$c^{n+1}_\alpha(\{i_1 < \cdots < i_{n+1}\}) = tp_\tau\left(f_{\beta_n(F_{n+1}(\alpha,n+1))} \circ \hat{g}^n_{F_{n+1}(\alpha,n+1)}(i_1, \ldots, i_{n+1})/\emptyset; M_{\beta_n(F_{n+1}(\alpha,n+1))}\right)$$

Because $\mathcal{K}^{n+1}$ is a combinatorial Erdős-Rado class, there are

$$\begin{aligned} \bar{h}^{n+1}_\alpha &: \bar{I}^{n+1}_\alpha \to \hat{I}^{n+1}_{F_{n+1}(\alpha,n+1)} \\ c^{*,n+1}_\alpha &: S^{\mathcal{K}^{n+1}}_{n+1,qf} \to S^\tau_{n+1,qf} \end{aligned}$$

such that $c^{*,n+1}_\alpha$ witnesses that $\bar{h}^{n+1}_\alpha(\bar{I}^{n+1}_\alpha)$ is type homogeneous for c^{n+1}_α.

Then finish as in Theorem 4.2. †

Thus, while we have no combinatorial partition result for $\mathcal{K}^{\omega-tr}$, it is an Erdős-Rado class.

Corollary 5.11. *$\mathcal{K}^{\omega-tr}$ is a cofinal Erdős-Rado class witnessed by $\beth_{(2^\mu)^+}$.*

Proof: By Theorem 5.10 applied to Proposition 5.9. †

6. Applications

6.1. **Unsuperstability in Abstract Elementary Classes.** In countable first-order theories, strict stability can detected by counting types at cardinals $\lambda < \lambda^\omega$: if T is stable, then T is superstable iff T is stable in some $\lambda < \lambda^\omega$ iff T is stable in all $\lambda < \lambda^\omega$. This is done by building what is called a 'Shelah tree' [Bal88, p. 85]. This is a way of embedding the $\omega+1$-heigh tree ${}^{\leq\omega}\lambda$ into a model of T so the types of branches are differentiated over their initial segments. In the context of atomic classes, Baldwin and Shelah [BS12, Theorem 3.3] generalized this to *atomic* classes by use of $\mathcal{K}^{\omega-tr}$-indiscernibles.

Here, we generalize this to tame Abstract Elementary Classes with amalgamation. Note that we break our convention of always using types over the empty set here. We say that $\mathbb{K}$ is Galois stable in μ iff $|S_{\mathbb{K}}(M)| \leq \mu$ for every $M \in \mathbb{K}_\mu$.

Theorem 6.1. *Let $\mathbb{K}$ be a $< \kappa$-tame Abstract Elementary Class with amalgamation. One of the following holds:*

1. *there is $\chi < \beth_{(2^{LS(\mathbb{K})})^+}$ such that $\mathbb{K}$ is Galois stable in every $\lambda > \chi$; or*
2. *$\mathbb{K}$ is Galois unstable in every $\lambda^\omega > \lambda$.*

Moreover, in case (2), there is $\Phi \in \Upsilon^{\omega-tr}[\mathbb{K}]$ such that $M_\lambda \prec EM_\tau({}^{<\omega}\lambda, \Phi)$ witnesses this.

The subscript '$\left(2^{\mathrm{LS}(\mathbb{K})}\right)^+$' can be replaced by the relevant undefinability of well-ordering number. This fits into the project summarized in [GV17]: while superstability for arbitrary Abstract Elementary Classes seems poorly behaved (exhibiting what Shelah terms 'schizophrenia'), superstability in the context of Abstract Elementary Classes is much better behaved. Vasey [Vas] computes stability spectra of Abstract Elementary Classes. For tame classes with amalgamation, [Vas, Corollary 4.24] uses a technical analysis of nonsplitting to show that failure of '$\mathbb{K}$ is Galois stable on a tail' implies $\chi(\mathbb{K}) > \omega$ and [Vas, Corollary 4.17] shows that for 'most' λ, cf $\lambda < \chi(\mathbb{K})$ implies $\mathbb{K}$ is Galois unstable in λ; 'most λ' means all sufficiently large, almost $\lambda(\mathbb{K})$-closed cardinals. Theorem 6.1 offers a tighter bound on when the tail of stability must start and also a better condition on where the instability must happen.

Our proof follows [BS12], but adapts the argument to Abstract Elementary Classes. The following notion of type fragments will make our argument smoother. These are essentially the partial Galois types that allow us to specify extending or not extending small Galois types. This is motivated by the idea that small Galois types should occasionally be able to stand in for formula in tame AECs (e.g., [Bon14, Section 3] or Vasey's Galois Morleyization [Vas16, Definition 3.3]).

Definition 6.2.

1. *Given $M \in \mathbb{K}$, $\mathcal{P}^*_\kappa M := \{M_0 \in \mathbb{K}_{<\kappa} : M_0 \prec M\}$.*
2. *A $< \kappa$-(Galois)* type fragment *over $M \in \mathbb{K}$ is a collection Σ of objects of the form 'p' or '$\neg p$' for $p \in S_{\mathbb{K}}(M_0)$ for $M_0 \in \mathcal{P}^*_\kappa M$ such that some element realizes every $p \in \Sigma$.*
3. *Some $a \in M$* realizes *a $< \kappa$-type fragment Σ over M iff $a \vDash p$ for all $p \in \Sigma$ and $a \nvDash p$ for all $\neg p \in \Sigma$.*
4. *A $< \kappa$-type fragment is* satisfiable *iff some element realizes it.*

We won't have use for unsatisfiable type fragments, so all type fragments will be assumed to be satisfiable.

The following is similar to an argument of Baldwin-Kueker-VanDieren (see [Bal09, Theorem 11.11]), generalizing first-order arguments of Morley. It will give us more than we need.

Lemma 6.3. *Suppose $N \in \mathbb{K}$ and $\Gamma \subset S_{\mathbb{K}}(N)$ has size greater than $\|N\|^{<\kappa}$. Then there is $M_1 \prec N$ of size $< \kappa$ and $r \neq q \in S_{\mathbb{K}}(M_1)$ such that both q and r have more than $\|N\|^{<\kappa}$-many extension to Γ.*

Proof: If not, then, for every $M_1 \in \mathcal{P}^*_\kappa N$, there is a unique $q_{M_1} \in S_{\mathbb{K}}(M_1)$ that has many extensions to Γ. Then every $p \in \Gamma$ is of one of two kinds:

1. $q \geq q_{M_0}$ for all $M_0 \in \mathcal{P}^*_\kappa N$; or
2. there is $M_0 \in \mathcal{P}^*_\kappa N$ such that $q \not\geq q_{M_0}$.

By tameness, there is at most one type in the first kind. By counting, there are at most $|\mathcal{P}^*_\kappa N| \times 2^{<\kappa} = \|N\|^{<\kappa}$ in the second kind. But Γ is too large, so we have a contradiction. †

Lemma 6.4. *Let $\mu > LS(\mathbb{K})$. If $M \in \mathbb{K}_{\geq 2^\mu}$ and $\langle p_\alpha \in S_{\mathbb{K}}(M) : \alpha < \|M\|^+\rangle$ are distinct, then there is $\langle N^i \in \mathcal{P}^*_\kappa M : i < \mu\rangle$ and $q(x; X) \in S^{<\kappa}_{\mathbb{K}}(\emptyset)$ such that one of the following occur:*

(1) *for all $j_1 < \mu$, the following set has size $\|M\|^+$*

$$\{i < \|M\|^+ : q(x; N^{j_1}) \not\leq p_i \text{ and } j_0 < j_1 \text{ implies } q(x; N^{j_0}) \leq p_i\}$$

(2) *for all $j_1 < \mu$, the following set has size $\|M\|^+$*

$$\{i < \|M\|^+ : q(x; N^{j_1}) \leq p_i \text{ and } j_0 < j_1 \text{ implies } q(x; N^{j_0}) \not\leq p_i\}$$

Proof: We build a tree $T \subset {}^{\leq\mu}2$; $\{q_\eta \in \mathrm{S}_{\mathbb{K}}(\emptyset) : \eta \in T\}$; and $\{N^\eta \in \mathcal{P}^*_\kappa M : \eta \in T\}$ such that:

(1) if $\eta \in T$, then the type fragment $\Sigma_\eta := \{q_{\eta\restriction j}(x; N^{\eta\restriction j})^{\eta(j)} : j < \ell(\eta)\}$ is contained in $> \|M\|$-many of the p_α's;
(2) if $i \leq \mu$ is limit, then

$$T_i := \{\eta \in {}^i2 : \Sigma_\eta \text{ is contained in } > \mu\text{-many of the } p_\alpha\text{'s and } \forall j < i, \eta \restriction j \in T\} \neq \emptyset$$

(3) if $i = j + 1$, then every node in T_j splits into two nodes.

This is enough: Pick $\eta \in T_\mu$. Set $q^j := q_{\eta\restriction j}$ for $j < \mu$. Since $\mu > 2^{\mathrm{LS}(\mathbb{K})}$, there is some $q \in \mathrm{S}_{\mathbb{K}}(\emptyset)$ such that $q = q^j$ for all $j \in X \in [\mu]^\mu$. Moreover, we can assume that $\eta(j) = \ell_\eta \in \{0, 1\}$ for all $j \in X$. If $\ell_\eta = 0$, we are in the first case; if $\ell_\eta = 1$, we are in the second case.

Construction: We work by induction on levels. There is nothing to do at $i = 0$. The successor step is precisely Lemma 6.3. At limit stage i, we note that every type p_α (or more generally, every $p \in \mathrm{S}_{\mathbb{K}}(M)$) extends one of our type fragments Σ_η for $\eta \in {}^i\mu$. There are μ^i many branches at this stage, and $> \|M\| > \mu^i$-many p_α's, so there must be some $\eta \in {}^i\mu$ so many of them extend Σ_η; then $\eta \in T_i$. †

Generalizing first-order notation, given $q \in \mathrm{S}_{\mathbb{K}}(M)$ and $p \in \mathrm{S}_{\mathbb{K}}(N)$, we write

$$q^0 \leq p \quad \text{to mean} \quad q \leq p$$
$$q^1 \leq p \text{ or } \neg q \leq p \quad \text{to mean} \quad q \not\leq p$$

The following lemma is the key inductive step that allows us to build our tree of types.

Lemma 6.5. *Suppose $M \in \mathbb{K}_{\geq 2^\mu}$ with $|\mathrm{S}_{\mathbb{K}}(M)| > \|M\|$, and let $\hat{M}$ be a μ^+-Galois saturated extension of M.*

There are increasing $\{M_n \in \mathbb{K}_\mu : n < \omega\}$; types $\{q_\nu \in S_{\mathbb{K}}(M_{\ell(n)}) : \nu \in {}^{<\omega}\mu\}$; models $\{N^\nu \in \mathcal{P}^*_\kappa M : \nu \in {}^{<\omega}\mu\}$; and elements $a_\nu \in \hat{M}$ realizing q_ν such that

(1) *each q_ν has $> \|M\|$-many extensions to $S_{\mathbb{K}}(M)$;*
(2) *if $n < m$ and $\nu \in {}^m\mu$, then*

$$q_{\nu\restriction n} \leq q_\nu$$

(3) *for every ν and $i \neq j$,*

$$q_{\nu^\frown\langle i\rangle} \neq q_{\nu^\frown\langle j\rangle}$$

moreover, they are different over $N^{\nu^\frown\langle\min(i,j)\rangle}$.

Proof: We do this by induction. Given stage n, we know that each q_ν for $\nu \in {}^n\mu$ has $> \|M\|$-many extensions to $\mathrm{S}_{\mathbb{K}}(M)$ and $\|M\| \geq 2^\mu$. So we can apply Lemma 6.4 to get $q_\nu \in \mathrm{S}^{<\kappa}_{\mathbb{K}}(\emptyset)$ and $\{N^i_\nu : i < \mu\}$ and $\ell_\nu = 0, 1$ such that for all $j_1 < \mu$, the following has size $\geq \|M\|^+$:

$$\{p \in \mathrm{S}_{\mathbb{K}}(M) : q_\nu \leq p \text{ and } q(x; N_\nu^{j_1})^{1-\ell_\nu} \leq p \text{ and } j_0 < j_1 \text{ implies } q(x; N_\nu^{j_0})^{\ell_\nu} \leq p\}$$

Set $N_{\nu^\frown\langle i\rangle} = N_\nu^i$.

Let $M_{n+1} \prec M$ contain M_n and $\bigcup_{\rho \in {}^{n+1}\mu} N_\rho$ of size μ. For each $i < \mu$, set

$$\Sigma'_{\nu,i} := q_\nu \cup \{q(x; N^i)^{1-\ell_\nu}, q(x; N^j)^{\ell_\nu} : j < i\}$$

This is a consistent type fragment over M by definition that can be extended to $> \|M\|$-many types over M. Using Lemma 6.3, we can extend this to a type $q_{\nu^\frown\langle i\rangle} \in \mathrm{S}_{\mathbb{K}}(M_{n+1})$ that can be extended to $> \|M\|$-many types over M. †

Proof of Theorem 6.1: Suppose that (1) fails, so that for $\alpha < \left(2^{\mathrm{LS}(\mathbb{K})}\right)^+$, there is $M^\alpha \in \mathbb{K}_{\geq \beth_{\alpha+1}}$ such that $|\mathrm{S}_{\mathbb{K}}(M_\alpha)| > \|M_\alpha\|$; by [Bon17, Theorem 3.1], we can assume this is witnessed by the 1-ary types. Using amalgamation, we can extend M^α to $\|M^\alpha\|^+$-Galois saturated $\hat{M}^\alpha$. Apply Lemma 6.5 for each α to get $\{M_n^\alpha, q_\nu^\alpha, N_\alpha^\nu, a_\nu^\alpha\}$ as there. Expand $\hat{M}^\alpha$ to M_+^α by

(1) adding the presentation language τ_1 to witness the $\prec$-relations and so there is $b_\alpha^\nu \in N_\alpha^\nu$ that generates N_α^ν;
(2) interpreting $\mathcal{K}^{\omega-tr}$-structure as follows:
   (a) P_n is M_n^α;
      (i) the presentation language should also be arranged so that M_n^α is generated by $\{b_\alpha^\nu : \nu \in {}^n\beth_\alpha\}$
   (b) $<$ and $\prec$ are the tree and lexicographic orderings on the b_α^ν (by their indices)
(3) R are the pairs (m, a_ν^α) for $m \in \bigcup_{n<\omega} M_n^\alpha$ and $\nu \in {}^{<\omega}\beth_\alpha$; and
(4) F_n takes b_α^ν to a_ν^α for $\nu \in {}^n\beth_\alpha$;
(5) S is a four place relation so that $S(\cdot, \cdot, b_\alpha^\nu, b_\alpha^\eta)$ defines the graph of a function that takes a_ν^α to a_η^α while fixing $P_{\ell(\nu\cap\eta)} = M_{\ell(\nu\cap\eta)}^\alpha$.

Then $|\tau_+| \leq \kappa$.

Since $\mathcal{K}^{\omega-tr}$ is a cofinal Erdős-Rado class (Corollary 5.11), we can build a blueprint $\Phi \in \Upsilon^{\omega-tr}[\mathbb{K}]$ that is modeled off of the embedding $\nu \in {}^{<\omega}\beth_\alpha \mapsto b_\alpha^\nu$this. Given $\chi \geq \kappa$, set T^χ to be the tree ${}^{<\omega}\chi$, $M_*^\chi = EM(T^\chi, \Phi)$, and $M^\chi := EM_\tau(T^\chi, \Phi)$. Clearly, $M^\chi \in \mathbb{K}_\chi$. Set $N^\chi := P^{M_*^\chi} \prec M^\chi$.

Claim: $|\mathrm{S}_{\mathbb{K}}(N^\chi)| \geq \chi^\omega$.

We will find distinct $p_\eta \in \mathrm{S}_{\mathbb{K}}(N^\chi)$ for $\eta \in {}^\omega\chi$.

Fix such an η. For $n < \omega$, set $p_\eta^n := \mathrm{gtp}\left(F_n^{M_*^\chi}(\eta \upharpoonright n)/P_n^{M_*^\chi}; M^\chi\right)$; recall our convention that $T^\chi \subset M^\chi$. By the S predicate, we have that $p_\eta^n \leq p_\eta^{n+1}$. By the ω-compactness of AECs with amalgamation, there is $p_\eta \in \mathrm{S}_{\mathbb{K}}(N^\chi)$ that extends all of these.

Suppose $\eta \neq \nu$. Then we can define

- $\rho = \eta \cap \nu$ and $i_\eta < i_\nu$ and $n = \ell(\rho)$;
- $\rho^\frown\langle i_\eta\rangle \leq \eta$ and $\rho^\frown\langle i_\nu\rangle \leq \nu$

Then, by construction,

$$p_\eta^{n+1} \upharpoonright N_*^{\rho^\frown\langle i_\eta\rangle} \neq p_\nu^{n+1} \upharpoonright N_*^{\rho^\frown\langle i_\eta\rangle}$$

where $N_*^{\rho^\frown\langle i_\eta\rangle}$ is the model generated by $\rho^\frown\langle i_\eta\rangle$. Thus the types are distinct. †

6.2. **Indiscernible Collapse in Nonelementary Classes.** One of the uses of generalized indiscernibles in first-order is to characterize various dividing lines via indiscernible collapse. An old result of Shelah [She90, Theorem II.2.13] says that a theory T is stable iff any order indiscernibles in a model of T are in fact set indiscernibles. Scow [Sco12, Theorem 5.11] proved that T is NIP iff any ordered graph indiscernibles in a model of T are in fact just order indiscernibles. In each of these cases, there is are abstract (Ramsey) classes $\mathcal{K}_0 \subset \mathcal{K}$ where some property of T can be detected by wether or not there are $\mathcal{K}$-indiscernibles that are not entirely determined by the smaller amount of information given by their $\mathcal{K}_0$-types. Guingona, Hill, and Scow [GHS] have formalized this notion of indiscernible collapse and given several more examples.

Following this work, we can give definitions of several dividing lines in Abstract Elementary Classes making use of the fact that the determining classes are Erdős-Rado classes in addition to being Ramsey classes. Unfortunately, at this time, we don't know of any indiscernible collapses characterizing dividing lines that start with an Erdős-Rado class (other than order indiscernibles, but this collapse result is already known for Abstract Elementary Classes). [GHS, Theorem 3.4] uses $\mathcal{K}^{n-mlo}$ and [GHS, Theorem 4.7] uses a class of trees that doesn't restrict the height (in a similar way that $\mathcal{K}^{ceq}$ generalizes $\mathcal{K}^{\chi-or}$), but neither of these are known to be Erdős-Rado classes. [GHS, Corollary 5.9] characterizes NTP_2 theories via a collapse of $\mathcal{K}^{ceq}$-indiscernibles, but involves notions of formulas dividing that does not easily generalize to Abstract Elementary Classes. This leads us to the following question:

Question 6.6. *Is $\mathcal{K}^{og}$ an Erdős-Rado class?*

Recall that it is consistently not a combinatorialy Erdős-Rado class by Example 3.16. However, this does not rule out the possibility it is an Erdős-Rado class. A positive answer for this question would give a prospective definition for the notion of NIP for Abstract Elementary Classes.

Definition 6.7. *Suppose that $\mathcal{K}^{og}$ is an Erdős-Rado class and let $\mathbb{K}$ be an Abstract Elementary Class with arbitrarily large models. We say that $\mathbb{K}$ is* NIP *iff for every $\Phi \in \Upsilon^{og}[\mathbb{K}]$, there is $\Psi \in \Upsilon^{or}[\mathbb{K}]$ such that $\Phi = \Psi \circ U$, where $U \in \Upsilon^{og}[\mathcal{K}^{or}]$ forgets the graph structure and $\Psi \circ U$ is the composition of these blueprints.*

This has advantages over other prospective definitions in that no amalgamation, tameness, etc. assumption is necessary. Of course, it has the disadvantage that it needs more results to be viable.

6.2.1. *Category theoretic interpretation of indiscernible collapse.* We can build on the category theoretic interpretation of indiscernibles from Section 5.2 to give the same gloss to indiscernible collapse in terms of injectivity conditions ([AR94, Section 4.A] gives this background). In general, if $f : A \to B$ is a morphism, then another object C is injective with respect to f iff every $g : A \to C$ can be lifted along f to a $g' : B \to C$ so $g = g' \circ f$. Then Shelah's result [She90,] can be rephrased as follows.

Theorem 6.8. *Let $\mathcal{K}^{set}$ be the abstract class of sets and $U : \mathcal{K}^{or} \to \mathcal{K}^{set}$ be the functor forgetting the ordering. An elementary class $\mathbb{K}$ is stable iff it is injective with respect to U (in the category of accessible categories whose morphisms are faithful functors preserving directed colimits).*

Proof: First, assume $\mathbb{K}$ is stable and let $G : \mathcal{K}^{or} \to \mathbb{K}$. By Theorem 5.5, we may assume that G comes from a blueprint Φ for order-indiscernibles. By [She90, Theorem II.2.13], Φ gives rise to a blueprint Ψ for set indiscernibles. Then, using Theorem 5.5 again, Ψ gives rise to the desired G'.

Second, suppose that $\mathbb{K}$ is injective in this sense. Then any order indiscernibles are set indiscernibles. By [She90, Theorem II.2.13], $\mathbb{K}$ is stable. †

Other indiscernible collapses can be phrased similarly.

6.3. **Interperability Order.** Finding generalized indiscernibles in nonelementary classes can give stronger negative results in the interpretability order even for comparing first-order theories. The interpretability order is a three-parameter order $\triangleleft^*_{\lambda,\chi,\kappa}$ on complete first-order theories introduced by Shelah [She96] in the vein of Keisler's order. It would say that T_0 is less complicated than T_1 iff every time a first-order theory interprets both T_0 and T_1, if the interpretation of T_1 is saturated, then so is the interpretation of T_0. [She96, Definition 2.10] gives the full definition, but we only need a particular instance, $\triangleleft^*_1$, which we strengthen to $\triangleleft^{*,\mu}_1$. Moreover, we allow the χ-parameter to be arbitrarily large (rather than countable as in most instances in [MS]) to strengthen our results. Since this application is not central, we omit some of the definitions, but a good exposition (and the results we reference) can be found in Malliaris and Shelah [MS].

Definition 6.9. *Let T_0 and T_1 be complete first-order theories and let μ be an infinite cardinal.*

1. *We say that $T_0 \triangleleft^*_1 T_1$ iff for all large enough, regular μ and for all infinite χ, there is a first-order theory T_* of size $\leq |T_0| + |T_1| + \chi$ that interprets T_ℓ via $\bar{\phi}_\ell$ such that, for every $M_* \vDash T_*$, if the interpretation $M_*^{[\bar{\phi}_1]}$ of T_1 is μ-saturated, then the $M_*^{[\bar{\phi}_0]}$ is μ-saturated. (Shelah)*
2. *We say that $T_0 \triangleleft^{*,\kappa}_1 T_1$ iff for all large enough, regular μ and for all infinite χ, there is an $\mathbb{L}_{\kappa,\omega}$-theory T_* of size $\leq |T_0| + |T_1| + \chi$ that interprets T_ℓ via $\bar{\phi}_\ell$ such that, for every $M_* \vDash T_*$, if the interpretation $M_*^{[\bar{\phi}_1]}$ of T_1 is μ-saturated, then the $M_*^{[\bar{\phi}_0]}$ is μ-saturated.*

So $\triangleleft^{*,\mu}_1$ differs from $\triangleleft^*_1$ in that it allows for infinitary theories to do the interpreting. In particular, the statement that $\neg(T_0 \triangleleft^{*,\mu}_1 T_1)$ is a stronger statement than $\neg(T_0 \triangleleft^*_1 T_1)$. In [MS], Malliaris and Shelah show several positive and negative instances of the interpretability order. The negative instances are proved by using various Ramsey classes to build generalized blueprints that saturate T_1 without saturating T_0. When these Ramsey classes are in fact Erdős-Rado classes, the stronger negative instance can be shown. In the following statement, T_{DLO} is the theory of dense linear orders and T_{RG} is the theory of the random graph.

Fact 6.10 ([MS, Theorem 5.3]). $\neg(T_{DLO} \triangleleft^*_1 T_{RG})$

Theorem 6.11. *For every cardinal κ, $\neg(T_{DLO} \triangleleft^{*,\kappa}_1 T_{RG})$.*

Proof: We rely heavily on citations from [MS]. Note their $\mathcal{K} = \mathcal{K}_{\S 4}$ is essentially our $\mathcal{K}^{\lambda-color}$, which is Erdős-Rado by Example 3.8 and Corollary 4.3. Also, they use GEM to emphasize that the Ehrenfeucht-Mostowski construction uses a generalized blueprint. We adopt [MS, Hypothesis 5.5] with our infinitary change, so

1. $\lambda = \lambda^{<\mu} \geq 2^\mu$;
2. T_* is a skolemized $\mathbb{L}_{\kappa,\omega}$-theory with $|T_*| \leq \lambda$ that interprets T_{RG} by R_{RG} and interprets T_{DLO} by $<_{DLO}$.

Note that they point out that their results in this area work for uncountable languages as well.

Since $\mathcal{K}$ is a combinatorial Erdős-Rado class, there is $\Phi \in \Upsilon^{\mathcal{K}}[T_*]$. By [MS, Corollary 5.10], we can find Ψ extending Φ such that for every separated $I \in \mathcal{K}$, $EM_{RG}(I, \Psi)$ is μ-saturated. Note that, since Ψ agrees with Φ on $\tau(T_*)$, Ψ is still in $\Upsilon^{\mathcal{K}}[T_*]$. By [MS, Claim 5.11], if J is a separated linear order, then for any $\Phi^* \in \Upsilon^{\mathcal{K}}[T_*]$, $EM_{DLO}(J, \Phi^*)$ is not κ^+-saturated. Thus, by taking I separated with a (κ, κ)-cut, we have $EM_{RG}(I, \Psi)$ is μ-saturated, but $EM_{DLO}(I, \Psi)$ is not κ^+-saturated, as desired. †

References

[AH78] Fred Abramson and Leo Harrington, *Models without indiscernibles*, Journal of Symbolic Logic **43** (1978), no. 3, 572–600.

[AR94] Jiří Adamek and Jiří Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Notes Series, no. 189, Cambridge University Press, 1994.

[Bal88] John Baldwin, *Fundamentals of stability theory*, Perspectives in Mathematical Logic, vol. 12, Springer-Verlag, 1988.

[Bal09] ______, *Categoricity*, University Lecture Series, no. 50, American Mathematical Society, 2009.

[Bod15] Manuel Bodirsky, *Ramsey classes: examples and constructions*, Surveys in Combinatorics 2015 (A. Czumaj, A. Georgakopoulos, D. Král, V. Lozin, and O. Pikhurko, eds.), London Mathematical Society Lecture Notes Series, Cambridge University Press, 2015, pp. 1–48.

[Bon] Will Boney, *Shelah's omitting types theorem*, Notes, `http://math.harvard.edu/~wboney/notes/SOTTNotes.pdf`.

[Bon14] ______, *Tameness from large cardinal axioms*, Journal of Symbolic Logic **79** (2014), no. 4, 1092–1119.

[Bon17] ______, *Computing the number of types of infinite length*, Notre Dame Journal of Formal Logic **58** (2017), no. 1, 133–154.

[BS12] John Baldwin and Saharon Shelah, *The stability spectrum for classes of atomic models*, Journal of Mathematical Logic **12** (2012), no. 1.

[Cha68] C. C. Chang, *Some remarks on the model theory of infinitary languages*, The Syntax and Semantics of Infinitary Languages (Jon Barwise, ed.), vol. 72, 1968, pp. 36–63.

[EHMR84] Paul Erdős, András Hajnal, Attila Máté, and Richard Rado, *Combinatorial set theory: Partition relations for cardinals*, Studies in Logic and the Foundations of Mathematics, vol. 106, North-Holland Publishing Company, 1984.

[EHR65] Paul Erdős, András Hajnal, and Richard Rado, *Partition relations for cardinal numbers*, Acta Mathematica **16** (1965), 93–196.

[EM56] Andrezej Ehrenfeucht and Andrzej Mostowski, *Models of axiomatic theories admitting automorphisms*, Fundamenta Mathematicae **43** (1956), no. 1, 50–68.

[ER56] Paul Erdős and Richard Rado, *A partition calculus in set theory*, Bulletin of the American Mathematical Society **62** (1956), 427–489.

[GHS] Vincent Guingona, Cameron Hill, and Lynn Scow, *Characterizing model-theoretic dividing lines via collapse of generalized indiscernibles*, Annals of Pure and Applied Logic, to appear.

[GS86] Rami Grossberg and Saharon Shelah, *A non-structure theorem for a theory which has the unsuperstability property*, Illinois Journal of Mathematics **30** (1986), 364–390.

[GS11] Estger Gruenhut and Saharon Shelah, *Uniforming n-place functions on well-founded trees*, Set Theory and Its Applications (L. Babinkostova, A. E. Caicedo, S. Geschke, and M. Scheepers, eds.), Contemporary Mathematics, vol. 533, American Mathematical Society, 2011, pp. 267–280.

[GV17] Rami Grossberg and Sebastien Vasey, *Equivalent definitions of superstability in tame abstract elementary classes*, Journal of Symbolic Logic **82** (2017), no. 4, 1387–1408.

[HK88] András Hajnal and Peter Komjáth, *Embedding graphs into colored graphs*, Transactions of the American Mathematical Society **307** (1988), no. 1, 395–409.

[HK92] ______, *Corrigendum to: "embedding graphs into colored graphs"*, Transactions of the American Mathematical Society **332** (1992), 475.

[HL10] András Hajnal and Jean Larson, *Partition relations*, Handbook of Set Theory (Matt Foreman and Akihiro Kanamori, eds.), Springer, 2010, pp. 129–213.

[Jec02] Thomas Jech, *Set thoery*, 3rd millenium ed., Springer-Verlag, 2002.

[KKS14] Bunghan Kim, Hyeung-Joon Kim, and Lynn Scow, *Tree indiscernibles, revisited*, Archive for Mathematical Logic **53** (2014), no. 1-2, 211–232.

[Lin69] Per Lindström, *On extensions of elementary logic*, Theoria **35** (1969), no. 1, 1–11.

[Mar02] David Marker, *Model theory: An introduction*, Graduate Texts in Mathematics, no. 217, Springer-Verlag, 2002.

[Mor65] Michael Morley, *Omitting classes of elements*, The theory of models, Proceedings of the 1963 International Symposium at Berkeley (John Addison, Leon Henkin, and Alfred Tarski, eds.), Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, 1965, pp. 265–273.

[MP89] Michael Makkai and Ropert Paré, *Accessible categories: the foundations of categorical model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society, 1989.

[MS] Maryanthe Malliaris and Saharon Shelah, *A new look at interpretability and saturation*, Submitted, `https://arxiv.org/abs/1709.04899`.

[MS90] Michael Makkai and Saharon Shelah, *Categoricity of theories in $\mathbb{L}_{\kappa,\omega}$, with κ a compact cardinal*, Annals of Pure and Applied Logic **47** (1990), 41–97.

[NR77] Jaroslav Nešetřil and Vojtěch Rödl, *A structural generalization of the ramsey theorem*, Bulletin of the American Mathematical Society **83** (1977), no. 1, 127–128.

[Ram30] Frank Plumpton Ramsey, *On a problem of formal logic*, Proceedings of the London Mathematical Society **30** (1930), no. 1, 264–286.

[Sco12] Lynn Scow, *Characterization of NIP theories by ordered graph-indiscernibles*, Annals of Pure and Applied Logic **163** (2012), 1624–1641.

[She] Saharon Shelah, *General non-struture theory*, `http://shelah.logic.at/nonstructure/III.pdf`, [ShE59], Chapter III of **Nonstructure theory**.

[She71] ______, *Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory*, Annals of Mathematical Logic **3** (1971), 271–362.

[She75] ______, *Generalized quantifiers and compact logic*, Transactions of the American Mathematical Society **204** (1975), 342–364.

[She87a] ______, *Classification of nonelementary classes, II. Abstract elementary classes*, Classification theory (John Baldwin, ed.), 1987, pp. 419–197.

[She87b] ______, *Universal classes*, Classification theory (John Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 264–418.

[She89] ______, *Consistency of positive partition theorems for graphs and models*, Set Theory and its applications (Toronto, ON, 1987), 1989, pp. 167–193.

[She90] ______, *Classification theory and the number of nonisomorphic models*, 2 ed., vol. 92, North-Holland Publishing Company, 1990.

[She96] ______, *Toward classifying unstable theories*, Annals of Pure and Applied Logic **80** (1996), 229–255.

[She99] ______, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), 261–294.

[She09] ______, *Classification theory for abstract elementary classes*, Mathematical Logic and Foundations, vol. 18 & 20, College Publications, 2009.

[TZ12] Katrin Tent and Martin Ziegler, *A course in model theory*, Lecture Notes in Logic, vol. 40, Cambridge University Press, 2012.

[Vas] Sebastien Vasey, *Toward a stability theory of tame abstract elementary classes*, Submitted, `https://arxiv.org/abs/1609.03252`.

[Vas16] ______, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), no. 3–4, 562–592.

E-mail address: `wboney@math.harvard.edu`

Department of Mathematics, Harvard University, Cambridge, MA, USA