DEFINABLE COHERENT ULTRAPOWERS AND ELEMENTARY EXTENSIONS

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Abstract. We develop the notion of coherent ultrafilters (extenders without normality or well-foundedness). We then use definable coherent ultraproducts to characterize any extension of a model M in any fragment of $\mathbb{L}_{\infty,\omega}$ that defines Skolem functions by a sufficiently complete (but in ZFC) coherent ultrafilter. We apply this method to various elementary classes and AECs.

1. Introduction

We combine the technique of definable ultrapowers and and coherent ultrafilters (a ZFC version of extenders) to characterize extension of a model in some fragment of $\mathbb{L}_{\infty,\omega}$ (including first order) with definable Skolem functions. Coherent ultrafilters are a weakening of extenders, a technique from set theory that first appear in Martin and Steel [MS89]. The goal of extenders is to capture some elementary embedding $j : V \to \mathcal{M}$ where $\mathcal{M}$ has some rank-bounded similarity to V by creating a coherent system of ultrafilters that satisfy additional properties (κ-completeness, normality, and well-foundedness; see Kanamori [Kan08] for a reference). The use of a coherent system of ultrafilters is necessary to separate out the properties closure under sequences (a saturation-like property) from containing, e. g., some V_λ. Rather than taking an ultrapower of V, the coherence gives a directed system of ultrapowers of V and the "extender power" (or coherent ultrapower in our terms) is the colimit of this system.

Definable ultrapowers are a technique that goes back to Skolem [Sko34] (although we use [Kei71, Chapter 32] as our more accessible reference). Rather than including *all* functions from the index set to the model M, the definable ultrapower uses some definable subset of M as the index and only considers the definable functions to the model. This has the drawback of requiring more structure on M, in particular, requiring $Th(M)$ to have definable Skolem functions. The pay-off, however, is that there is much greater control over what the definable ultrapower looks like. The work in [Kei71, Chapter 32] exploits this to, from $M \prec N$, create a ZFC ultrafilter that is complete enough to ensure not just first-order elementarity of the extension by the definable ultrapower, but elementarity according to whatever fragment of $\mathbb{L}_{\infty,\omega}$ that captures the relation between M and N. The key result that we generalize is the following.

Fact 1.1 ([Kei71]). *Let $\mathcal{F}$ be a fragment of $\mathbb{L}_{\infty,\omega}(\tau)$ and $\psi \in \mathcal{F}$ have definable Skolem functions. Given $M \prec_{\mathcal{F}} N$ that model ψ, there is an $\mathcal{F}$-complete ultrafilter U on M such the definable ultrapower $\prod^{def} M/U$ $\mathcal{F}$-elementary embeds into N over M. Moreover, given some $c \in N$, U can be constructed so that c is in the image of this embedding.*

The moreover is the key to our combination. The coherent ultrafilters allow us to move from specifying some $c \in N$ in the target model to specifying an arbitary subset of the target model. This allows us to make $\prod^{def} M/U$ isomorphic to N (see Theorem 3.5). In order to do this,

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we must show that the $\mathcal{F}$-complete coherent filter derived from $M \prec_{\mathcal{F}} N$ can be extended to a $\mathcal{F}$-complete coherent ultrafilter; this is the goal of Section 2. In Section 4, we work through some examples of classes and apply this theorem.

Note that, in the context of $(V, \in)$ every function $f : \kappa \to V$ is definable with parameters[1], so the distinction between $\prod V/E$ and $\prod^{def} V/E$ disappears. A few notes on notation are in order. We refer to a system of ultrafilters that is coherent (see Definition 2.1) as a "coherent ultrafilter" rather than the clunkier but more accurate "coherent system of ultrafilters." We also discuss some general facts about coherent ultraproducts.

After this paper was first circulated, Enayat, Kauffman, and McKenzie circulated [EKM], which also deals with coherent ultraproducts (although they call them 'dimensional ultrapowers' and crucially restrict to a single ultrafilter). Their work focuses on exposition (containing exercises and solutions) and using coherent ultrapowers to build tight indiscernibles. [EKM, Section 4] also contains a nice overview of the history and applications for the interested reader.

2. Coherent Ultrafilters

The following notion of coherent ultrafilter is designed to take the key model-theoretic features of an extender (see [Kan08, Chapter 26]) that allow the construction of $\prod V/E$, and remove the parts that correspond to beyond-ZFC strength. We allow the arity μ of the extender to be larger than ω (that is, a (κ, λ)-extender is a $(\omega, \kappa, \lambda)$-coherent ultrafilter with extra structure), although we will not use it here. Also, we remove the requirement that κ be a cardinal, replacing it with an arbitrary set A. Although this does not matter for the full coherent ultrapower since there is a bijection between A and $|A|$, it is an important consideration for the definable coherent ultrapower since the bijection is rarely definable.

We use a slightly different formalism for coherent filters than typical. Normally, for $\mathbf{a} \in [\lambda]^{<\omega}$, one of the following is used:

- $F_{\mathbf{a}}$ is a filter on $[A]^{|\mathbf{a}|}$ with A, so there is a canonical pairing between $\mathbf{a}$ and any $s \in [A]^{|\mathbf{a}|}$ that pairs each element of $\mathbf{a}$ with the corresponding one of s (this is used in [Kan08])
- $F_{\mathbf{a}}$ is a filter on ${}^{\mathbf{a}}A$, so, for $s \in {}^{\mathbf{a}}A$, the pairing of elements is given by the function (this is used in [MS89])

The pairing is necessary to define the projections $\pi^{\mathbf{b}}_{\mathbf{a}}$ for $\mathbf{a} \subset \mathbf{b}$, which is how one makes sense of coherence. Since we want to work with definable objects in some model, neither of these are available. Instead, we will take $\mathbf{a} \in [\lambda]^{<\omega}$ and set $F_{\mathbf{a}}$ to be a filter on ${}^{|\mathbf{a}|}A$. Then we can pair the ith element of $\mathbf{a}$ with the ith element of $s \in {}^{|\mathbf{a}|}A$ in the order inherited from the function. This is clunkier, but is necessitated by our situation. As part of this, $s \in {}^{n}A$ will be considered both as an n-tuple of elements and as a function from n to A.

Definition 2.1. *Let A be a set, $\mu \leq \lambda$ cardinals. $[A]^{<\mu}$ is the collection of $< \mu$-sized subsets of A and $(\lambda)^{<\mu}$ are the subsets of λ with order-type $< \mu$.*

(1) *Let $\alpha < \beta < \mu$, $\mathbf{a} \in (\lambda)^{\alpha}$, $\mathbf{b} \in (\lambda)^{\beta}$. Set $p_{\mathbf{a},\mathbf{b}} : otp(\mathbf{a}) \to otp(\mathbf{b})$ be the unique order-preserving injection such that the ith member of $\mathbf{a}$ is the $p_{\mathbf{a},\mathbf{b}}(i)$th member of $\mathbf{b}$. Set $\pi^{\mathbf{b}}_{\mathbf{a}} : {}^{\beta}A \to {}^{\alpha}A$ by*
$$\pi^{\mathbf{b}}_{\mathbf{a}}(s) = s \circ p_{\mathbf{a},\mathbf{b}}$$
(2) *We say that F is a (μ, A, λ)-*coherent filter *iff $F = \{F_{\mathbf{a}} \mid \mathbf{a} \in (\lambda)^{<\mu}\}$ such that*
  (a) *each $F_{\mathbf{a}}$ is a filter on ${}^{otp(\mathbf{a})}A$; and*
  (b) *if $\mathbf{a} \subset \mathbf{b}$, then for any $X \subset {}^{otp(\mathbf{a})}A$, we have*
$$X \in F_{\mathbf{a}} \iff \{s \in {}^{otp(\mathbf{b})}A \mid \pi^{\mathbf{b}}_{\mathbf{a}}(s) \in X\} \in F_{\mathbf{b}}$$

[1] By the formula "$x = f$."

(3) *We say that* E *is a* (μ, A, λ)-coherent ultrafilter *iff it is a* (μ, A, λ)*-coherent filter with each* $E_{\mathbf{a}}$ *an ultrafilter.*
(4) *A* (μ, A, λ)*-coherent filter* F *is* proper *iff no* $F_{\mathbf{a}}$ *contains the empty set.*
(5) *Given two* (μ, A, λ)*-coherent filters* F *and* F^**, we say that* F^* extends F *or* $F \Subset F^*$ *iff* $F_{\mathbf{a}} \subseteq F^*_{\mathbf{a}}$ *for all* $\mathbf{a} \in (\lambda)^{<\mu}$*.*
(6) *Given* $\mathbf{b} \subset \mathbf{c}$ *and* $X \subset {}^{otp(\mathbf{c})}A$*, we say that* X *is full over* $\mathbf{b}$ *iff for every* $s, t \in {}^{otp(\mathbf{c})}A$ *such that* $\pi^{\mathbf{c}}_{\mathbf{b}}(s) = \pi^{\mathbf{c}}_{\mathbf{b}}(t)$*, we have*

$$s \in X \iff t \in X$$

(7) *If we omit* μ *from any of the above definitions, we mean* $\mu = \omega$*.*

We deal almost exclusively with the case $\mu = \omega$. In this case, we write $[\lambda]^{<\omega}$ instead of $(\lambda)^{<\omega}$ and $|\mathbf{a}|$ instead of $\mathrm{otp}(\mathbf{a})$; they are the same, and the former follows convention. We could also vary this definition to allow different $F_{\{a\}}$ to live on different sets, rather than the same one; this is sometimes necessary with extenders. However, since we don't impose extra conditions like normality, we can extend the underlying set so all $F_{\{a\}}$ live on the same one in the context of coherent filters.

Given $\mathbf{a} \subset \mathbf{b} \subset \mathbf{c}$ and $X \subset {}^{|\mathbf{b}|}A$, we give $\pi^{\mathbf{b}}_{\mathbf{a}} X$ and $(\pi^{\mathbf{c}}_{\mathbf{b}})^{-1} X$ the natural meanings. Fullness is a useful notion because it allows us to write the coherence condition as the conjunction of the following:

- If $\mathbf{a} \subset \mathbf{b}$ and $X \in F_{\mathbf{a}}$, then $(\pi^{\mathbf{b}}_{\mathbf{a}})^{-1} X \in F_{\mathbf{b}}$.
- If $\mathbf{a} \subset \mathbf{b}$ and $X \in F_{\mathbf{b}}$ is full over $\mathbf{a}$, then $\pi^{\mathbf{b}}_{\mathbf{a}} X \in F_{\mathbf{a}}$.

This is used in, e. g., Lemma 2.6.

Proposition 2.2. *Given* $\mathbf{a} \subset \mathbf{b}$*, each subset* X *of* ${}^{|\mathbf{b}|}A$ *has a minimal extension* X^+ *that is full over* $\mathbf{a}$*, sometimes called the fullification of* X *over* $\mathbf{a}$*. This set has the property that* $\pi^{\mathbf{b}}_{\mathbf{a}} X = \pi^{\mathbf{b}}_{\mathbf{a}} X^+$*.*

Proof: Straightforward. †

The goal of this section is to generalize the fact that any nonprincipal filter can be extended to a nonprincipal ultrafilter to coherent filters and ultrafilters; this is Theorem 2.7. This seems trickier than normal since adding a new set to some $F_{\mathbf{b}}$ commits new sets to the rest of the filters (by coherence) and could break the nonprincipality of some $F_{\mathbf{a}}$. However, as the following lemmas show, the original coherence is enough to ensure this doesn't happen.

The first lemma is a technical calculation that will come in handy later.

Lemma 2.3. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in [\lambda]^{<\mu}$ *with* $\mathbf{b} \subset \mathbf{c}$*,* $X \subset {}^{otp(\mathbf{c})}A$*, and* $Y \subset {}^{otp(\mathbf{a})}A$ *such that* X *is full over* $\mathbf{b}$ *and*

$$(\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} Y \subset X$$

Set Y^+ *to be the fullification of* Y *over* $\mathbf{a} \cap \mathbf{b}$*. Then*
(1) $(\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} Y^+ \subset X$*; and*
(2) $\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} Y^+$ *is full over* $\mathbf{a} \cap \mathbf{b}$*.*

In this proof and others, we define elements of, e. g., ${}^{\mathrm{otp}(\mathbf{c})}A$. The notation we use makes this difficult to parse, but writing, for instance,

$$s'(i) = s\left(p^{-1}_{\mathbf{b},\mathbf{c}}(i)\right) \text{ if defined}$$

means: if the ith element of $\mathbf{c}$ occurs in $\mathbf{b}$ as the jth element (so $p_{\mathbf{b},\mathbf{c}}(j) = i$), then evaluate $s'(i)$ to $s(j)$.

Proof: For (1), let $s \in (\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} Y^+$. Unravelling the definition, this means there is $t \in Y^+$ and $t_0 \in Y$ such that

- $\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}} s = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} t$; and
- $\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} t = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} t_0$.

Define $s' \in {}^{\text{otp}(\mathbf{c})}A$ by

$$s'(i) = \begin{cases} s\left(p^{-1}_{\mathbf{b},\mathbf{c}}(i)\right) & \text{if defined} \\ t_0\left(p^{-1}_{\mathbf{a}\cap\mathbf{c},\mathbf{c}}(i)\right) & \text{if defined} \\ arb. & \text{o/w} \end{cases}$$

Note that

$$\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{b}}(s) = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}}(t) = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}}(t_0)$$

so this is well defined function. Then $\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}}(s') = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}}(t_0) \in \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} Y$, so $s' \in X$. Also, $\pi^{\mathbf{c}}_{\mathbf{b}}(s') = \pi^{\mathbf{c}}_{\mathbf{b}}(s)$ and X is full over $\mathbf{b}$, so $s \in X$, as desired.

For (2), let $s \in \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} Y^+$ and $s' \in {}^{\text{otp}(\mathbf{a}\cap\mathbf{c})}A$ such that $\pi^{\mathbf{a}\cap\mathbf{c}}_{\mathbf{a}\cap\mathbf{b}}(s) = \pi^{\mathbf{a}\cap\mathbf{c}}_{\mathbf{a}\cap\mathbf{b}}(s')$. From the first inclusion, there is $t \in Y^+$ such that $s = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}}(t)$. Define $t' \in {}^{|\mathbf{a}|}A$ by

$$t'(i) = \begin{cases} s'\left(p^{-1}_{\mathbf{a}\cap\mathbf{c},\mathbf{a}}(i)\right) & \text{if defined} \\ t\left(p^{-1}_{\mathbf{b},\mathbf{a}}(i)\right) & \text{if defined} \\ arb. & \text{o/w} \end{cases}$$

Since $\pi^{\mathbf{a}\cap\mathbf{c}}_{\mathbf{a}\cap\mathbf{b}}(s') = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} t$, this is well-defined. Also, $\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}}(t) = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}}(t')$ and $t \in Y^+$ is full over $\mathbf{a}\cap\mathbf{b}$, so $t' \in Y^+$. Thus, $s' = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}}(t') \in \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} Y^+$, as desired. †

The following lemma tells us that it's enough to push new sets down and then up.

Lemma 2.4. *Suppose* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in (\lambda)^{<\mu}$ *such that* $\mathbf{a} \subset \mathbf{c}$ *and* $\mathbf{b} \subset \mathbf{c}$. *Let* $X \subset {}^{otp(\mathbf{a})}A$ *be full over* $\mathbf{a}\cap\mathbf{b}$. *Then*

$$(\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1}\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X = \pi^{\mathbf{c}}_{\mathbf{b}}(\pi^{\mathbf{c}}_{\mathbf{a}})^{-1} X$$

Proof: First, let $s \in (\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1}\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X$. Then there is $t \in X$ such that $\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}}(s) = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}}(t)$. Define $s' \in {}^{\text{otp}(\mathbf{c})}A$ by

$$s'(i) = \begin{cases} s\left(p^{-1}_{\mathbf{b},\mathbf{c}}(i)\right) & \text{if defined} \\ t\left(p^{-1}_{\mathbf{a},\mathbf{c}}(i)\right) & \text{if defined} \\ arb. & \text{o/w} \end{cases}$$

We have that this is well-defined. From definition, we have $\pi^{\mathbf{c}}_{\mathbf{a}}(s') = t \in X$ and $\pi^{\mathbf{c}}_{\mathbf{b}}(s') = s$. Thus, $s \in \pi^{\mathbf{c}}_{\mathbf{b}}(\pi^{\mathbf{c}}_{\mathbf{a}})^{-1} X$.

Second, let $x \in \pi^{\mathbf{c}}_{\mathbf{b}}(\pi^{\mathbf{c}}_{\mathbf{a}})^{-1} X$. Then there is $s_0 \in {}^{\text{otp}(\mathbf{c})}A$ and $t \in X$ such that $\pi^{\mathbf{c}}_{\mathbf{b}}(s_0) = s$ and $\pi^{\mathbf{c}}_{\mathbf{a}}(s_0) = t$. This gives

$$\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}}(s) = \pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{b}}(s_0) = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} t$$

Thus, $s \in (\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1}\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X$. †

Lemma 2.5. *Suppose* F *is a* (μ, A, λ)*-coherent filter and* $F^*_{\mathbf{a}}$ *is a proper filter on* ${}^{otp(\mathbf{a})}A$ *extending* $F_{\mathbf{a}}$. *Then, for all* $\mathbf{b} \in (\lambda)^{<\mu}$,

$$F_{\mathbf{b}} \cup \{(\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1}\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X \mid X \in F^*_{\mathbf{a}} \text{ is full over } \mathbf{a}\cap\mathbf{b}\}$$

has the finite intersection property.

Proof: Suppose that this is false for some $\mathbf{b} \in (\lambda)^{<\mu}$. Then there are $Y \in F_{\mathbf{b}}$ and $X_1, \ldots, X_n \in F^*_{\mathbf{a}}$ that are full over $\mathbf{a} \cap \mathbf{b}$ such that

$$Y \cap \bigcap_i (\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X_i = \emptyset$$

Set $X = \cap_i X_i \in F^*_{\mathbf{a}}$. Then we have

$$Y \cap (\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X = \emptyset$$

Set Y^+ to be the fullification of Y over $\mathbf{a} \cap \mathbf{b}$. Then $Y^+ \in F_{\mathbf{b}}$ and $(\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}} Y^+ \in F_{\mathbf{a}}$. Since $F^*_{\mathbf{a}}$ is proper, there is

$$t \in (\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}} Y^+ \cap X$$

Thus $t \in X$ and $\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}}(t) = \pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}} t_0$ for some $t_0 \in Y^+$. By definition of Y^+, there is $t_1 \in Y$ such that $\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}}(t) = \pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}} t_1$. This means that $t_1 \in (\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X$.

Thus, $t_1 \in Y \cap (\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X$, a contradiction. †

Lemma 2.6. *Let F be a (μ, A, λ)-coherent filter and $F^*_{\mathbf{a}}$ a proper filter on ${}^{otp(\mathbf{a})}A$ extending $F_{\mathbf{a}}$. Set $F^*_{\mathbf{b}}$ to be the filter generated by*

$$F_{\mathbf{b}} \cup \{(\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X \mid X \in F^*_{\mathbf{a}} \textit{ is full over } \mathbf{a} \cap \mathbf{b}\}$$

Then F^* is a proper (μ, A, λ)-coherent filter.

Proof: For each $\mathbf{b} \in (\lambda)^{<\mu}$, $F^*_{\mathbf{b}}$ is a proper filter by Lemma 2.5. Thus we must make sure the system is coherent. Fix $\mathbf{b} \subset \mathbf{c} \in (\lambda)^{<\mu}$.

First, suppose that $X \in F^*_{\mathbf{b}}$ and we want to show that $(\pi^{\mathbf{c}}_{\mathbf{b}})^{-1} \in F^*_{\mathbf{c}}$. From the definition of $F^*_{\mathbf{b}}$, there is $X \in F_{\mathbf{b}}$ and $X_1, \ldots, X_n \in F^*_{\mathbf{a}}$ full over $\mathbf{a} \cap \mathbf{b}$ such that

$$Y \cap \bigcap_i (\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X_i \subset X$$

Applying $(\pi^{\mathbf{c}}_{\mathbf{b}})^{-1}$ to both sides and noting $(\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} = (\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}}$, we have

$$(\pi^{\mathbf{c}}_{\mathbf{b}})^{-1} Y \cap \bigcap (\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} X_i \subset (\pi^{\mathbf{c}}_{\mathbf{b}})^{-1} X$$

Thus, $(\pi^{\mathbf{c}}_{\mathbf{b}})^{-1} X \in F^*_{\mathbf{c}}$, as desired.

Second, suppose that $X \in F^*_{\mathbf{c}}$ is full over $\mathbf{b}$ and we want to show $\pi^{\mathbf{c}}_{\mathbf{b}} X \in F^*_{\mathbf{b}}$. From the definition of $F^*_{\mathbf{c}}$, there is $Y \in F_{\mathbf{c}}$ and $X_1, \ldots, X_n \in F^*_{\mathbf{a}}$ full over $\mathbf{a} \cap \mathbf{c}$ such that

$$Y \cap \bigcap_i (\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} X_i \subset X$$

Using Lemma 2.3, we can assume that Y is full over $\mathbf{b}$ and each X_i is full over $\mathbf{a} \cap \mathbf{b}$; this implies $\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} X_i$ is full over $\mathbf{a} \cap \mathbf{b}$. The goal is to show that

$$\pi^{\mathbf{c}}_{\mathbf{b}} Y \cap \bigcap_i (\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X_i \subset \pi^{\mathbf{c}}_{\mathbf{b}} X$$

Since F was a coherent filter, this suffices to show $\pi^{\mathbf{c}}_{\mathbf{b}} X \in F^*_{\mathbf{b}}$. Let $s \in \pi^{\mathbf{c}}_{\mathbf{b}} Y \cap \bigcap_i (\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}} X_i$. By definition, this means there are $t \in Y$ and $s_i \in X_i$ such that $s = \pi^{\mathbf{c}}_{\mathbf{b}}(t)$ and $\pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}}(s) = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}}(s_i)$.

We claim that $t \in (\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} X_i$. To see this, we can compute that

$$\pi^{\mathbf{a}\cap\mathbf{c}}_{\mathbf{a}\cap\mathbf{b}} \left(\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}}(t)\right) = \pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}} \left(\pi^{\mathbf{c}}_{\mathbf{b}}(t)\right) = \pi^{\mathbf{b}}_{\mathbf{a}\cap\mathbf{b}}(s) = \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{b}}(s_i) \in \pi^{\mathbf{a}\cap\mathbf{c}}_{\mathbf{a}\cap\mathbf{b}} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} X_i$$

Since $\pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} X_i$ is full over $\mathbf{a} \cap \mathbf{c}$, this gives $\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}}(t) \in \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} X_i$, as desired.

Thus, $t \in Y \cap \bigcap_i (\pi^{\mathbf{c}}_{\mathbf{a}\cap\mathbf{c}})^{-1} \pi^{\mathbf{a}}_{\mathbf{a}\cap\mathbf{c}} X_i$. By assumption, we have $t \in X$ and $s = \pi^{\mathbf{c}}_{\mathbf{b}}(t) \in \pi^{\mathbf{c}}_{\mathbf{b}} X$, as desired. †

This brings us to the main theorem.

Theorem 2.7. *Any coherent filter can be extended to a coherent ultrafilter.*

Proof: We use Zorn's Lemma. Let F be a coherent filter and $\mathcal{Z}$ be the collection of coherent filters extending F ordered by $\Subset$. Clearly, the union of any $\Subset$-increasing chain of coherent filters is a coherent filter, so Zorn's Lemma says there is a maximal element E. If some $E_{\mathbf{a}}$ is not an ultrafilter, then there is a proper filter $E^*_{\mathbf{a}}$ extending it. By Lemma 2.6, this gives rise to $E^* \Supset E$, contradicting it's maximality. Thus, E is a coherent ultrafilter. †

Theorem 2.7 suffices for Section 3, where we deal with definable coherent ultrapowers. For completeness, we include some basic facts about coherent ultrafilters and coherent ultraproducts. For the rest of the section, we focus on (μ, κ, λ)-coherent ultrafilters. Since we don't deal with definability, there is no loss. First note that Theorem 2.7 does not prove the existence of proper coherent ultrafilter (although this follows by Theorem 3.5 applied to any proper extensions). The following result says that we can generate a coherent ultrafilter starting with any collection of "seeds" (at least when $\mu = \omega$).

Theorem 2.8. *Let $\{U_\alpha \mid \alpha < \lambda\}$ be a collection of ultrafilters on κ. Then there is an $(\omega, \kappa, \lambda)$-coherent ultrafilter E such that, for all $\alpha < \lambda$, $E_{\{\alpha\}}$ is U_α after passing through the canonical bijection from ${}^1\kappa$ to κ.*

For this proof, recall the product of filters: if F_ℓ is a filter on I_ℓ for $\ell = 0, 1$, then $F_0 \otimes F_1$ is the filter on $I_0 \times I_1$ given by

$$X \in F_0 \otimes F_1 \iff \{i \in I_0 \mid \{j \in I_1 \mid (i,j) \in X\} \in F_1\} \in F_0$$

$F_0 \otimes F_1$ is a filter, is an ultrafilter iff F_0 and F_1 are, and the productive is associative, although non-commutative. We use this product slightly modified to our situation so, e. g., the product of a filter on ${}^n\kappa$ and a filter on ${}^m\kappa$ is a filter on ${}^{n+m}\kappa$.

Proof: We define E by induction by setting, for $\mathbf{a} \in [\lambda]^{<\omega}$:

(1) If $\mathbf{a} = \{\alpha\} \in [\lambda]^1$, then $E_{\mathbf{a}} = U_\alpha$.
(2) If $\mathbf{a} \in [\lambda]^n$ is $\alpha_0 < \cdots < \alpha_{n-1}$, then $E_{\mathbf{a}}$ is the common value of

$$U_{\alpha_0} \otimes E_{\{\alpha_1, \dots, \alpha_{n-1}\}} = E_{\{\alpha_0, \dots, \alpha_{i-1}\}} \otimes U_{\alpha_i} \otimes E_{\{\alpha_{i+1}, \dots, \alpha_{n-1}\}} = E_{\{\alpha_0, \dots, \alpha_{n-2}\}} \otimes U_{\alpha_{n-1}}$$

E is clearly a collection of ultrafilters, so it only remains to show coherence. It is enough to check coherence for one-point extensions.

Let $\mathbf{b} \in [\lambda]^{n+1}$ be $\alpha_0 < \cdots < \alpha_n$, $X \subset {}^n\kappa$, and $i < n+1$. Set $\mathbf{a} = \mathbf{b} - \{\alpha_i\}$ and $X^* = \{s \in {}^{n+1}\kappa \mid \pi^{\mathbf{b}}_{\mathbf{a}}(s) \in X\}$. For notational ease, set $\mathbf{a}_1 = \{\alpha_0, \dots, \alpha_{i-1}\}$ and $\mathbf{a}_2 = \{\alpha_{i+1}, \dots, \alpha_n\}$. We have

$$\begin{aligned}
X \in E_{\mathbf{a}} &\iff X \in E_{\mathbf{a}_1} \otimes E_{\mathbf{a}_2} \\
&\iff \left\{s \in {}^i\kappa \mid \{t \in {}^{n-i}\kappa \mid s^\frown t \in X\} \in E_{\mathbf{a}_2}\right\} \in E_{\mathbf{a}_1} \\
X^* \in E_{\mathbf{b}} &\iff X \in E_{\mathbf{a}_1} \otimes U_{\alpha_i} \otimes E_{\mathbf{a}_2} \\
&\iff \left\{s \in {}^i\kappa \mid \left\{j \in {}^1\kappa \mid \{t \in {}^{n-i}\kappa \mid s^\frown j^\frown t \in X\} \in E_{\mathbf{a}_2}\right\} \in U_{\alpha_i}\right\} \in E_{\mathbf{a}_1}
\end{aligned}$$

We have $\pi^{\mathbf{b}}_{\mathbf{a}}(s^\frown j^\frown t) = s^\frown t$, so $s^\frown t \in X$ iff $s^\frown j^\frown t \in X^*$. This finishes the proof. †

Most applications of coherent ultrafilters (and extenders) involve taking the coherent relative of the ultrapower–which we call the coherent ultrapower–of a single model. This is also true of our Theorem 3.5. However, one can naturally define a coherent ultraproduct. In fact, there are at least two notions of a coherent ultraproduct one might define; one is more general than the other, but this extra generality borders on "too general" and there seems to be no application of it that doesn't reduce to a simple case (yet).

First, the less general. Let E be an (μ, κ, λ)-coherent ultrafilter, $\mathbf{a} \in (\lambda)^\alpha$, and $\{M_s \mid s \in {}^\alpha\kappa\}$ be a collection of τ-structures. The *coherent ultraproduct of* $\{M_s \mid s \in {}^\alpha\kappa\}$ *by* E *at* $\mathbf{a}$ is denoted by $\prod^{\mathbf{a}} M_s/E$ and is constructed as follows: for each $\mathbf{b} \in {}^\beta\lambda$ that extends $\mathbf{a}$, form the standard ultraproduct

$$M_*^{\mathbf{b}} := \prod_{s \in {}^\beta\kappa} M_{\pi^{\mathbf{b}}_{\mathbf{a}}(s)}/E_{\mathbf{b}}$$

Then if $\mathbf{c} \in (\lambda)^{<\mu}$ extends $\mathbf{b}$ (and therefore also $\mathbf{a}$), there is a natural map $f^{\mathbf{b},\mathbf{c}} : M_*^{\mathbf{b}} \to M_*^{\mathbf{c}}$ by taking $[f]_{E_{\mathbf{b}}}$ to $[f \circ \pi^{\mathbf{c}}_{\mathbf{b}}]_{E_{\mathbf{c}}}$. Łoś' Theorem shows that $f^{\mathbf{b},\mathbf{c}}$ is an elementary embedding. This is a directed system, and we take the colimit to form $\prod^{\mathbf{a}} M_s/E$. The coherent ultrapower of M by E is this construction with $M_s = M$. This means that a coherent ultrapower is simultaneously a coherent ultraproduct at $\mathbf{a}$ for every $\mathbf{a} \in (\lambda)^{<\mu}$. Note that the isomorphism type of the coherent ultraproduct is independent of the ordering on λ (in the sense that any permutation of λ induces an automorphism of coherent ultraproducts).

We can generalize this further by imposing the minimum structure necessary to make this construction work. Fix an (μ, κ, λ)-ultrafilter E and structures $\{M_s^{\mathbf{a}} \mid \mathbf{a} \in (\lambda)^{<\mu}, s \in {}^{\text{otp}(\mathbf{a})}\kappa\}$ such that if $\mathbf{b}$ extends $\mathbf{a}$ and $t \in {}^{\text{otp}(\mathbf{b})}\kappa$, then $\tau\left(M^{\mathbf{a}}_{\pi^{\mathbf{b}}_{\mathbf{a}}(t)}\right) \subset \tau(M_t^{\mathbf{b}})$ and there is elementary $f^{\mathbf{a},\mathbf{b}}_{\pi^{\mathbf{b}}_{\mathbf{a}}(t),t} : M^{\mathbf{a}}_{\pi^{\mathbf{b}}_{\mathbf{a}}(t)} \to M_t^{\mathbf{b}}$. Then we can again form the ultraproducts $M_*^{\mathbf{a}} = \prod_{s \in {}^{|\mathbf{a}|}\kappa} M_s^{\mathbf{a}}/E_{\mathbf{a}}$. If $\mathbf{b}$ extends $\mathbf{a}$, then the $f^{\mathbf{a},\mathbf{b}}_{s,t}$ induce $f^{\mathbf{a},\mathbf{b}}_* : M_*^{\mathbf{a}} \to M_*^{\mathbf{b}}$ by taking $[f]_{E_{\mathbf{a}}}$ to $[t \mapsto f^{\mathbf{a},\mathbf{b}}_{\pi^{\mathbf{b}}_{\mathbf{a}}(s),s}\left(f(\pi^{\mathbf{b}}_{\mathbf{a}}(s))\right)]_{E_{\mathbf{b}}}$. Again, this is elementary by Łoś' Theorem and the elementarity of each $f^{\mathbf{a},\mathbf{b}}_{s,t}$. Set $\prod M_s^{\mathbf{a}}/E$ to be the colimit of this system.

We can view coherent ultraproducts at some $\mathbf{a}$ as these more general coherent ultraproducts by setting

$$M_s^{\mathbf{b}} = \begin{cases} M_{\pi^{\mathbf{b}}_{\mathbf{a}}(s)} & \mathbf{b} \text{ extends } \mathbf{a} \\ \emptyset & \text{otherwise} \end{cases}$$

and each $f^{\mathbf{a},\mathbf{b}}_{s,t}$ is the identity. However, we know of no use for this extra level of generality. As further evidence for their strangeness, suppose E was a (κ, λ)-extender and $j_E : V \to M_E$ was the derived embedding. Then an coherent ultraproduct at $\mathbf{a}$ (for any $\mathbf{a} \in [\lambda]^{<\omega}$) appears in M_E: if f is the functions that takes $s \in {}^{|\mathbf{a}|}\kappa$ to M_s, then $\prod M_s/E \cong j(f)\left(j^{-1} \restriction j(\mathbf{a})\right)$. However, the more general coherent ultraproducts don't seem to appear.

3. The Definable Coherent Ultrapower

We define the notion of a definable coherent ultrapowers, which combines the notion of a coherent ultraproduct and a definable ultraproduct (see [Kei71]).

Fix some fragment[2] $\mathcal{F} \subset \mathbb{L}_{\infty,\omega}(\tau)$ containing at least all atomic formulae and a sentence $\psi \in \mathcal{F}$. We are going to work with the class $\mathcal{K} = (\text{Mod } \psi, \prec_{\mathcal{F}})$. Note that this includes first order classes (take $\mathcal{F}$ to be $\mathbb{L}_{\omega,\omega}(\tau)$ along with a single conjunction for the theory). The following definitions are natural generalizations of those in [Kei71, Chapter 32].

Definition 3.1. *Fix* $\psi(z) \in \mathcal{F}$.

(1) *A definable function* $\phi(\mathbf{x}, y)$ *with domain* $\chi(z)$ *is a formula* $\phi(\mathbf{x}, y) \in \mathcal{F}$ *such that*

$$\psi \vDash \forall \mathbf{x} \left(\wedge_i \chi(x_i) \to \exists^{=1} y \phi(\mathbf{x}, y)\right)$$

We sometimes write this function as $F_{\phi(\mathbf{x})}$.

[2] We define a fragment to be a collection of formulas that is closed under subformula; this is a weaker notion than the one in [Kei71, Chapter 4].

(2) *Let* $\chi(z) \in \mathcal{F}$. *We say that* $\mathcal{K}$ *has* definable Skolem functions over $\chi(z)$ *iff for every* $\exists y\sigma(\mathbf{x}, y) \in \mathcal{F}$, *there is a definable function* $F_{\exists y\sigma(\mathbf{x},y)}$ *with domain* $\chi(z)$ *such that*

$$\psi \vDash \forall \mathbf{a} \left(\bigwedge_i \chi(a_i) \wedge \exists b\sigma(\mathbf{a}, b) \to \sigma\left(\mathbf{a}, F_{\exists y\sigma(\mathbf{x},y)}(\mathbf{a})\right)\right)$$

(3) *Suppose that* $\mathcal{K}$ *has definable Skolem functions over* $\chi(z)$. *Let* $M \in \mathcal{K}$ *and* E *be a* $(\omega, \chi(M), \lambda)$*-coherent ultrafilter for some* λ.

(a) *Set* $\prod^{def}_{\chi,n} M$ *to be set of all* n*-ary definable functions with parameters with domain* $\chi(z)$.

(b) *For* $F, G \in \prod^{def}_{\chi,n} M$ *and* $\mathbf{a} \in [\lambda]^n$, *set*

$$F \sim_{E_{\mathbf{a}}} G \iff \{s \in {}^n\chi(M) \mid M \vDash \text{“}F(s) = G(s)\text{”}\} \in E_{\mathbf{a}}$$

(c) *Set* $\prod^{def}_{\chi,n} M/E_{\mathbf{a}}$ *to be the model with universe* $\{[F]_{E_{\mathbf{a}}} \mid F \in \prod^{def}_{\chi,n} M\}$ *with the standard ultraproduct structure, e. g., if* $R \in \tau$, *then*

$$R^{\prod^{def}_{\chi,n} M/E_{\mathbf{a}}}([F_1]_{E_{\mathbf{a}}}, \dots, [F_n]_{E_{\mathbf{a}}}) \iff \{s \in {}^n\chi(M) \mid M \vDash R\left(F_1(s), \dots, F_n(s)\right)\} \in E_{\mathbf{a}}$$

(d) *Given* $\mathbf{a} \in [\lambda]^n$ *and* $\mathbf{b} \in [\lambda]^m$ *such that* $\mathbf{a} \subset \mathbf{b}$, *define* $f_{\mathbf{a},\mathbf{b}} : \prod^{def}_{\chi,n} M/E_{\mathbf{a}} \to \prod^{def}_{\chi,m} M/E_{\mathbf{b}}$ *by*

$$f_{\mathbf{a},\mathbf{b}}\left([F]_{E_{\mathbf{a}}}\right) = [F \circ \pi^{\mathbf{b}}_{\mathbf{a}}]_{E_{\mathbf{b}}}$$

(e) *The definable coherent ultrapower of* M *by* E *on* $\chi(z)$ *is the directed colimit of the sequence*

$$\left\{\prod^{def}_{\chi,n} M/E_{\mathbf{a}}, f_{\mathbf{a},\mathbf{b}} \mid \mathbf{a} \subset \mathbf{b} \in [\lambda]^{<\omega}\right\}$$

And is denoted

$$\left(\prod^{def}_{\chi} M/E, f_{\mathbf{a},\infty}\right)_{\mathbf{a} \in [\lambda]^{<\omega}}$$

(4) *In the above definitions, if* $\chi(z) \equiv$ “$z = z$”, *then we omit it, e. g., writing* $\prod^{def} M/E$.

We use $\mu = \omega$ here because there are not $\mathbb{L}_{\infty,\omega}$-definable functions with infinite arity. However, this theory generalizes to the construction of definable $(\mu, \chi(M), \lambda)$-coherent ultraproducts in $\mathbb{L}_{\infty,\mu}$-axiomatizable classes (or μ-AECs; see Boney, Grossberg, Lieberman, Rosicky, and Vasey [BGL+16], especially the Presentation Theorem 3.2 there).

During the definition, we assumed that certain definitions were well-defined. We note this now.

Proposition 3.2. *The construction of* $\prod^{def}_{\chi} M/E$ *is well defined; that is,*

(1) $\sim_{E_{\mathbf{a}}}$ *is a* τ*-congruence relation on* $\prod^{def}_{\chi,n} M$*; and*
(2) *given* $\mathbf{a} \subset \mathbf{b} \subset \mathbf{c} \in [\lambda]^{<\omega}$, *we have that* $f_{\mathbf{a},\mathbf{b}}$ *is a* τ*-embedding* $f_{\mathbf{a},\mathbf{c}} = f_{\mathbf{b},\mathbf{c}} \circ f_{\mathbf{a},\mathbf{b}}$.

Proof: The proof is a straightforward calculation. †

It will be useful to recall the form of any element in a $(\mu, \chi(M), \lambda)$-coherent ultrapower: given $x \in \prod^{\text{def}}_{\chi} M/E$, $x = f_{\mathbf{a},\infty}\left([F]_{E_{\mathbf{a}}}\right)$ for some $\mathbf{a}$ and F, where F is a $|\mathbf{a}|$-ary definable function from $\chi(M)$ to M; in fact, there are many such $\mathbf{a}$ and F. We will write this as $[\mathbf{a}, F]_E$. Then it is easy to check that

$$[\mathbf{a}, F]_E = [\mathbf{b}, G]_E \iff \{s \in {}^{|\mathbf{a}\cup\mathbf{b}|}\chi(M) \mid M \vDash F \circ \pi^{\mathbf{a}\cup\mathbf{b}}_{\mathbf{a}}(s) = G \circ \pi^{\mathbf{a}\cup\mathbf{b}}_{\mathbf{b}}(s)\} \in E_{\mathbf{a}\cup\mathbf{b}}$$

One final notion is necessary to show that this construction interacts well with the (potentially) non-elementary class $\mathcal{K}$. Typically, highly complete (and therefore non-ZFC) ultrafilters are necessary to preserve $\mathbb{L}_{\infty,\omega}$ formulas. However, since we only deal with *definable* functions, we have more leeway.

Definition 3.3.

(1) *Suppose $M \in \mathcal{K}$, $\phi(x_1, \dots, x_n) \in \mathcal{F}$, and $f_1, \dots, f_m$ are n-ary definable functions with domain $\chi(M)$. Then set*

$$\phi \circ (f_1, \dots, f_m) (\chi(M)) := \{s \in {}^n\chi(M) \mid M \vDash \phi (f_1(s), \dots, f_m(s))\}$$

(2) *For $M \in \mathcal{K}$, we say that a filter F on ${}^n\chi(M)$ is $\mathcal{F}$-complete iff for all $\wedge_{\alpha<\kappa}\phi_\alpha(x_1, \dots, x_m) \in \mathcal{F}$ and n-ary definable functions $f_1, \dots, f_m$ (with respect to ψ and with parameters) with domain $\chi(M)$, if, for all $\alpha < \kappa$, $\phi_\alpha \circ (f_1, \dots, f_m) (\chi(M)) \in F$, then $\wedge_{\alpha<\kappa}\phi_\alpha \circ (f_1, \dots, f_m) (\chi(M)) \in F$.*

(3) *We say that a coherent filter F is $\mathcal{F}$-complete iff each $F_{\mathbf{a}}$ is.*

Proposition 3.4. *Suppose $\mathcal{K}$ has definable Skolem functions over $\chi(z)$, $M \in \mathcal{K}$, and E is an $\mathcal{F}$-complete $(\omega, \chi(M), \lambda)$-coherent ultrafilter.*

(1) *For all $\mathbf{a} \subset \mathbf{b} \in [\lambda]^{<\omega}$, the embeddings $f_{\mathbf{a},\mathbf{b}}$ and $f_{\mathbf{a},\infty}$ are $\mathcal{F}$-elementary.*

(2) *Let $[\mathbf{a}_1, F_1]_E, \dots, [\mathbf{a}_n, F_n]_E \in \prod_\chi^{def} M/E$ and $\phi(x_1, \dots, x_n) \in \mathcal{F}$. Then*

$$\prod_\chi^{def} M/E \vDash \phi ([\mathbf{a}_1, F_1]_E, \dots, [\mathbf{a}_n, F_n]_E)$$

iff

$$\phi \circ (F_1 \circ \pi_{\mathbf{a}_1}^{\cup \mathbf{a}_i}, \dots, F_n \circ \pi_{a_n}^{\cup \mathbf{a}_i}) (\chi(M)) \in E_{\cup \mathbf{a}_i}$$

(3) $\prod_\chi^{def} M/E \in \mathcal{K}$

(4) *The coherent ultrapower embedding $j : M \to \prod_\chi^{def} M/E$ is $\mathcal{F}$-elementary and is also proper iff some $E_{\mathbf{a}}$ is non-principal.*

We will refer to item (2) as Łoś' Theorem for definable coherent ultraproducts. Note that the set appearing there is

$$\left\{s \in {}^{|\cup \mathbf{a}_i|}\chi(M) \mid M \vDash \phi \left(F_1(\pi_{\mathbf{a}_1}^{\cup \mathbf{a}_i}(s)), \dots, F_n(\pi_{\mathbf{a}_n}^{\cup \mathbf{a}_i}(s))\right)\right\}$$

In the proof, we will use the fact that, if F is a definable function, then so is $F \circ \pi_{\mathbf{a}}^{\mathbf{b}}$.

Proof: 1) We will use Łoś' Theorem for definable ultraproducts, see [Kei71, Theorem 46].

Suppose that $\mathbf{a} \subset \mathbf{b} \in [\lambda]^{<\omega}$ and $[F_1]_{E_{\mathbf{a}}}, \dots, [F_k]_{E_{\mathbf{a}}} \in \prod_{\chi,n}^{\text{def}} M/E_{\mathbf{a}}$. Let $\phi(x_1, \dots, x_k) \in \mathcal{F}$. Then

$$\prod_{\chi,n}^{\text{def}} M/E_{\mathbf{a}} \vDash \phi ([F_1]_{E_{\mathbf{a}}}, \dots, [F_k]_{E_{\mathbf{a}}}) \iff \phi \circ (F_1, \dots, F_k) (\chi(M)) \in E_{\mathbf{a}}$$

$$\iff (\pi_{\mathbf{a}}^{\mathbf{b}})^{-1}\phi \circ (F_1, \dots, F_k) (\chi(M)) = \phi \circ (F_1 \circ \pi_{\mathbf{a}}^{\mathbf{b}}, \dots, F_k \circ \pi_{\mathbf{a}}^{\mathbf{b}}) (\chi(M)) \in E_{\mathbf{b}}$$

$$\iff \prod_{\chi,m}^{\text{def}} M/E_{\mathbf{b}} \vDash \phi (f_{\mathbf{a},\mathbf{b}}([F_1]_{E_{\mathbf{a}}}), \dots, f_{\mathbf{a},\mathbf{b}}([F_k]_{E_{\mathbf{a}}}))$$

We used the coherence of E in the second line, and the definition of $f_{\mathbf{a},\mathbf{b}}$ in the final line. The $\mathcal{F}$-elementarity of the $f_{\mathbf{a},\infty}$ follows from the $\mathcal{F}$-elementarity of the $f_{\mathbf{a},\mathbf{b}}$ since $\mathcal{F}$-elementary maps are closed under directed colimits; this is the crucial use of $\mathbb{L}_{\infty,\omega}$ as compared with $\mathbb{L}_{\infty,\infty}$.

2) This follows from (1) and the fact that $[\mathbf{a}_i, F_i]_E = f_{\mathbf{a},\infty}([F_i]_{E_{\mathbf{a}_i}}) = f_{\cup\mathbf{a}_j,\infty}([F_i \circ \pi^{\cup\mathbf{a}_j}_{\mathbf{a}_i}]_{E_{\cup\mathbf{a}_j}}$. Then

$$\prod_{\chi}^{\text{def}} M/E \vDash \phi\left([\mathbf{a}_1, F_1]_E, \dots, [\mathbf{a}_n, F_n]_E\right)$$

$$\iff \prod_{\chi,m}^{\text{def}} M/E_{\cup\mathbf{a}_i} \vDash \phi\left([F_1 \circ \pi^{\cup\mathbf{a}_i}_{a_1}]_{E_{\cup a_i}}, \dots, [F_n \circ \pi^{\cup\mathbf{a}_i}_{\mathbf{a}_n}]_{E_{\cup\mathbf{a}_i}}\right)$$

$$\iff \phi \circ (F_1 \circ \pi^{\cup\mathbf{a}_i}_{\mathbf{a_1}}, \dots, F_n \circ \pi^{\cup\mathbf{a}_i}_{\mathbf{a}_n})\,(\chi(M)) \in E_{\cup\mathbf{a}_i}$$

3) $\psi \in \mathcal{F}$, so this follows from (1).

4) For $\mathbf{a} \in [\lambda]^{<\omega}$ and $m \in M$, set $F^{\mathbf{a}}_m$ to be the constant m function with domain ${}^{\mathbf{a}}\chi(M)$; this is definable with the parameter m. Then $[\mathbf{a}, F^{\mathbf{a}}_m]_E = [\mathbf{b}, F^{\mathbf{b}}_m]_E$ for all $\mathbf{a}, \mathbf{b} \in [\lambda]^{<\omega}$.

Define j to take $m \in M$ to $[\mathbf{a}, F^{\mathbf{a}}_m]_E$ for some/any $\mathbf{a} \in [\lambda]^{<\omega}$. Then j is elementary by Łoś' Theorem.

For the properness, first suppose that $[\mathbf{a}, G]_E \in \prod^{\text{def}}_{\chi} M/E$ is not in the range of j. If $E_{\mathbf{a}}$ is principal, then it is generated by some $s \in {}^{|\mathbf{a}|}\chi(M)$. Then $[\mathbf{a}, G]_E = [\mathbf{a}, F^{\mathbf{a}}_{G(s)}]_E$, contradicting $[\mathbf{a}, G]_E$ not being in the range of j. Second, suppose that some $E_{\mathbf{a}}$ is non-prinicipal. Let $\text{id}_{\mathbf{a}}$ be the identity function on ${}^{|\mathbf{a}|}\chi(M)$. Then $[\mathbf{a}, \text{id}_{\mathbf{a}}]_E \notin j(M)$. †

This shows that the existence of a $\mathcal{F}$-complete coherent ultrafilter will give rise to an extension. Our main theorem is that the converse holds as well.

Theorem 3.5. *Suppose that $\mathcal{K}$ has definable Skolem functions over $\chi(z)$ and $M \prec_{\mathcal{F}} N$ from $\mathcal{K}$. Let $\lambda = |\chi(N)|$ and $f : \lambda \to \chi(N)$ be a bijection. Then there is an $\mathcal{F}$-complete $(\omega, \chi(M), \lambda)$-coherent ultrafilter E and $\mathcal{F}$-elementary $h : \prod^{def}_{\chi} M/E \to N$ such that $h \circ j = id_M$ and*

$$h\text{"}\chi\left(\prod_{\chi}^{def} M/E\right) = \chi(N)$$

Proof: For $\mathbf{a} \in [\lambda]^n$, define $F_{\mathbf{a}}$ to be the filter generated by sets of the form $\phi\circ(F_1, \dots, F_m)\,(\chi(M))$ for $\phi(x_1, \dots, x_m) \in \mathcal{F}$ and $F_1, \dots, F_m \in \prod^{\text{def}}_{\chi,n} M$ such that $N \vDash \phi\left(F_1(f\text{"}\mathbf{a}), \dots, F_m(f\text{"}\mathbf{a})\right)$.

Claim 3.6. *F is an $\mathcal{F}$-complete $(\omega, \chi(M), \lambda)$-coherent filter.*

Proof: Each $F_{\mathbf{a}}$ is a filter by definition (and will be principal exactly when $f\text{"}\mathbf{a} \in \chi(M) \subset \chi(N)$). Nearly by definition, each $F_{\mathbf{a}}$ is $\mathcal{F}$-complete.

To show coherence, let $\mathbf{a} \subset \mathbf{b} \in [\lambda]^{<\omega}$. We use the characterization following Definition 2.1.

First, if $X \in F_{\mathbf{a}}$, then it contains some $\phi\circ(F_1, \dots, F_m)\,(\chi(M))$ such that $N \vDash \phi\left(F_1(f\text{"}\mathbf{a}), \dots, F_m(f\text{"}\mathbf{a})\right)$. Then $(\pi^{\mathbf{b}}_{\mathbf{a}})^{-1}X$ contains $\phi \circ (F_1 \circ \pi^{\mathbf{b}}_{\mathbf{a}}, \dots, F_m \circ \pi^{\mathbf{b}}_{\mathbf{a}})\,(\chi(M))$ and

$$N \vDash \phi\left(F_1\left(f\text{"}\pi^{\mathbf{b}}_{\mathbf{a}}(\mathbf{b})\right), \dots, F_m\left(f\text{"}\pi^{\mathbf{b}}_{\mathbf{a}}(\mathbf{b})\right)\right)$$

Thus, $(\pi^{\mathbf{b}}_{\mathbf{a}})^{-1}X \in F_{\mathbf{b}}$.

Second, suppose $X \in F_{\mathbf{b}}$ is full over $\mathbf{a}$ and it contains $\phi \circ (F_1, \dots, F_k)\,(\chi(M))$. We can write F_i as $F_i(x_1, \dots, x_n; y_1, \dots, y_m)$, where the x_i's correspond to $\mathbf{a}$ and the y_i's correspond to $\mathbf{b} - \mathbf{a}$. Fix $\mathbf{m} \in {}^{|\mathbf{b}|}\chi(M)$ to be made of definable constants and set $F^*_i(\mathbf{x}) = F_i(\mathbf{x}, \mathbf{m})$. Note that $F^*_i \in \prod^{\text{def}}_{\chi,n} M$.

Claim 3.7. $\phi \circ (F^*_1, \dots, F^*_k)\,(\chi(M)) \subset \pi^{\mathbf{b}}_{\mathbf{a}}X$

Proof: Let $s \in \phi \circ (F_1^*, \ldots, F_k^*)(\chi(M))$. Thus, $M \vDash \phi(F_1(s; \mathbf{m}), \ldots, F_k(s; \mathbf{m}))$. So

$$s \cup \mathbf{m} \in \phi \circ (F_1, \ldots, F_k)(\chi(M)) \subset X$$

$$s = \pi^{\mathbf{b}}_{\mathbf{a}}(s \cup \mathbf{m}) \in \pi^{\mathbf{b}}_{\mathbf{a}} X$$

$\dagger_{\text{Claim 3.7}}$

Then $\pi^{\mathbf{b}}_{\mathbf{a}} X \in F_{\mathbf{b}}$. $\dagger_{\text{Claim 3.6}}$

Now, by Theorem 2.7, there is an $(\omega, \chi(M), \lambda)$-coherent ultrafilter E extending F. Since all of the relevant sets–the $\phi \circ (f_1, \ldots, f_n)(\chi(M))$–are already in F, the $\mathcal{F}$-completeness of F transfers to E.

Thus, we can form $\prod_{\chi}^{\text{def}} M/E$ and $\mathcal{F}$-elementary $j : M \to \prod_{\chi}^{\text{def}} M/E$ by Proposition 3.4. We build a system of maps $\{h_{\mathbf{a}} : \prod_{\chi,n}^{\text{def}} M/E_{\mathbf{a}} \to N \mid \mathbf{a} \in [\lambda]^{<\omega}\}$ as in [Kei71, Corollary following Theorem 47]: set $h_{\mathbf{a}}([F]_{E_{\mathbf{a}}}) = F^N(f\text{"}\mathbf{a})$. The same argument as [Kei71] shows that this is a well defined $\mathcal{F}$-embedding. It is also a straightforward calculation to show that, for any $\mathbf{a} \subset \mathbf{b} \in [\lambda]^{<\omega}$,

$$h_{\mathbf{a}} = h_{\mathbf{b}} \circ f_{\mathbf{a},\mathbf{b}}$$

Then we can push the maps through the colimit to get $\mathcal{F}$-elementary $h_\infty : \prod_{\chi}^{\text{def}} M/E \to N$ such that $h_{\mathbf{a}} = h_\infty \circ f_{\mathbf{a},\infty}$; this has an explicit description

$$h_\infty([\mathbf{a}, F]_E) = F^N(f\text{"}\mathbf{a})$$

Finally, we show that h_∞ is surjective. Let $n \in N$. Clearly, the identity function is definable, so

$$h_\infty\left([\{f^{-1}(n)\}, \text{id}]_E\right) = n$$

$\dagger_{\text{Theorem 3.5}}$

This gives a nice corollary that characterizes $\mathcal{F}$-elementary extensions by coherent ultrafilters.

Corollary 3.8. *Suppose that $\mathcal{K}$ has definable Skolem functions and $M \prec_{\mathcal{F}} N$ from $\mathcal{K}$. Then there is an $\mathcal{F}$-complete $(\omega, |M|, \|N\|)$-coherent ultrafilter E such that N and $\prod^{def} M/E$ are isomorphic via a map extending j.*

Note that this is stronger than related results in set theory, e. g., an extender derived from a strong cardinal can only capture an initial segment of the target model, while Corollary 3.8 captures the entire target model. This comes from the fact that our target model N is a set, while the target model of a strong embedding is a proper class.

4. Examples

4.1. **First-order.** We apply this theory to some elementary classes with definable Skolem functions. Here, $\mathcal{F}$ can be taken to be $\mathbb{L}_{\omega,\omega}$ (or, pedantically, the smallest fragment of $\mathbb{L}_{|T|^+,\omega}$ containing $\wedge_{\phi \in T}\phi$, but those are equivalent). This simplifies matters as the property of a coherent ultrafilter being $\mathcal{F}$-complete is vacuously satisfied.

The easiest way to have definable Skolem functions is via Skolemization. Thus, we can apply Corollary 3.8 to achieve the following:

Corollary 4.1. *Let T be a first order theory, $M \vDash T$, and M_{Sk} and T_{Sk} be the Skolemizations. Then any $N \succ M$ can be expanded to $N_{Sk} \succ M_{Sk}$ and there is a $(\omega, |M|, \|N\|)$-coherent ultrafilter E such that $\prod^{def} M_{Sk}/E \restriction \tau(M)$ and N are isomorphic over M.*

However, the Skolemization requires extra information and $\prod^{\text{def}} M_{Sk}/E \restriction \tau(M)$ is very different from $\prod^{\text{def}} M/E$ (the latter of which might not be a structure). Thus we examing a few cases with definable Skolem functions.

4.1.1. *Peano Arithmetic.* PA has definable Skolem functions and a prime, minimal model $\mathbb{N}$ [CK12, Example 3.4.5]. This means that every model of PA can be viewed (up to isomorphism) as an extension of $\mathbb{N}$ and, using Theorem 3.5, can be characterized by a $(\omega, \omega, \lambda)$-coherent ultrafilter.

Let $M \succ \mathbb{N}$. Fix some bijection $f : \lambda \to |M|$. Define F^M to be the $(\omega, \omega, \lambda)$-coherent filter defined by, for $\mathbf{a} \in [\lambda]^n$, $F^M_{\mathbf{a}}$ is all subsets of $\mathbb{N}$ that contains some $\phi(\mathbb{N})$ such that $M \vDash \phi(f"\mathbf{a})$. This can be extended to a $(\omega, \omega, \lambda)$-coherent ultrafilter E. Define $h : M \to \prod^{\text{def}} \mathbb{N}/E$ as in Theorem 3.5 to take $m \in M$ to $[\{f^{-1}(m)\}, \text{id}]_E$. This is an isomorphism.

4.1.2. *o-minimal Theories and Transseries.* A more complex example can be made from any o-minimal theory that expands a group since all such theories have definable Skolem functions [vdD98]. As an application of this, one could take the restriction of transseries $\mathbb{T}$ (see [Avd-DvdH17]) to the language of ordered fields and view this as a extension of $\mathbb{R}$. Then, we can write $\mathbb{T}$ as a definable ultrapower of $\mathbb{R}$, which encodes it into the combinatorial of a coherent ultrafilter $E^{\mathbb{T}}$ on $\mathbb{R}$.

Other examples of first order theories with definable Skolem functions include p-adically closed fields and real closed domains [vdD84, Theorem 3.2 and Corollary 3.6].

4.2. **Beyond first-order.** Now we explore a general method of applying this construction to nonelementary classes. We use Abstract Elementary Classes (AECs) as our framework of choice since they encompass many others. This is not always ideal (the 'Skolem functions' from Shelah's Presentation Theorem are not always the best possible), and we discuss other cases below. For this section, we assume that the reader is familiar with AECs and point them towards Baldwin [Bal09] as a reference.

Let $\mathcal{K} = (\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC with $\kappa = \text{LS}(\mathcal{K})$. Given $M \prec_{\mathcal{K}} N$, we would like to realize N as a definable coherent ultrapower of M. However, we have no notion of definability, so this is difficult (to say the least). Thus, we must appeal to Shelah's Presentation Theorem [She87, Conclusion 1.13] (see also [Bal09, Theorem 4.15]). This says that we can represent $\mathcal{K}$ as follows:

- $\tau^* := \tau(\mathcal{K}) \cup \{F^i_n(\mathbf{x}) \mid n < \omega, i < \kappa\}$;
- $T^* = \{\forall \mathbf{x} F^k_n(\mathbf{x}) = x_k \mid k < n < \omega\}$ is a (first-order) τ^*-theory;
- Γ is a set of quantifier-free τ^*-types;
- the models of $\mathcal{K}$ are precisely the $\tau(\mathcal{K})$-reducts of models of T^* omitting each type in Γ (this is written $PC(T^*, \Gamma, \tau(\mathcal{K}))$); and
- for $M, N \in \mathcal{K}$, we have $M \prec_{\mathcal{K}} N$ iff there are expansions M^* and N^* of M and N to models of T^* that omit Γ and $M^* \subset N^*$.

Set

$$\psi := \bigwedge_{p \in \Gamma} \forall x_1, \ldots, x_n \bigvee_{\phi \in p} \neg\phi(\mathbf{x})$$

and $\mathcal{F}$ to be the smallest fragment of $\mathbb{L}_{(2^\kappa)^+, \omega}(\tau^*)$ containing ψ that is closed under finite conjunctions. Note that the definable functions of this language are just the functions of the language and that it trivially has definable Skolem functions (because $\mathcal{F}$ has no existentials).

Now suppose that $M \prec_{\mathcal{K}} N$. Then we can expand these to τ^*-structures satisfying ψ such that $M^* \subset N^*$. From the definition of the fragment, this gives $M^* \prec_{\mathcal{F}} N^*$. Thus, we can use Theorem 3.5 to find an $\mathcal{F}$-complete $(\omega, |M|, \|N\|)$-coherent ultrafilter E such that $\prod^{def} M^*/E \cong N^*$.

Restricting to the original language, we have

$$\left(\prod^{def} M^*/E\right) \restriction \tau(K) \cong N$$

In general, we have that the ultraproduct operation commutes with the restriction of languages. However, this is not the case with *definable* ultraproducts because the notion of wether or not a function is definable is very language dependent.

The expansion to the Shelah Presentation is very coarse in that it pays no attention to how the original AEC was defined. We might have even started with a very nice first-order class with built-in Skolem functions, and then made it more complex by adding extra functions. Working with the original definition can often lead to nicer Skolem functions.

Recall that $\mathbb{L}(Q)$ is first-order logic with an additional quantifier Q that stands for "there exists uncountably many." If T is a $\mathbb{L}(Q)(\tau)$-theory and $\mathcal{F} \subset \mathbb{L}(Q)(\tau)$ is a fragment containing T, then we define $\prec^*_{\mathcal{F}}$ by

$$\begin{aligned} M \prec^*_{\mathcal{F}} N \quad \iff \quad & M \prec_{\mathcal{F}} N \text{ and if } \mathbf{a} \in M \text{ and } M \vDash Qx\phi(x, \mathbf{a}), \\ & \text{then } \phi(N, \mathbf{a}) = \phi(M, \mathbf{a}) \end{aligned}$$

where $\prec_{\mathbb{L}(Q)}$ is elementary substructure according to the logic, then $\mathcal{K} := (\text{Mod } T, \prec^*)$ is an AEC with $\text{LS}(\mathcal{K}) = \aleph_1 + |T|$ (see [Bal09, Exercise 5.1.3]). We wish to add Skolem function to this class. One option is to use Shelah's Presentation Theorem. A more precise option is to do the following:

- $\tau_{Sk} = \tau \cup \{F_\phi(\mathbf{y}) \mid \exists x\phi(x, \mathbf{y}) \in \mathcal{F}\} \cup \{G^{0,\alpha}_\phi(\mathbf{y}) \mid Qx\phi(x, \mathbf{y}) \in \mathcal{F}, \alpha < \omega_1\} \cup \{G^{1,n}_\phi(\mathbf{y}) \mid \neg Qx\phi(x, \mathbf{y}) \in \mathcal{F}, n < \omega\}$
- T_{Sk} is T with the following changes:
  1. every place "$\exists x\phi(x, \mathbf{y})$" appears, replace it with "$\phi(F_\phi(\mathbf{y}), \mathbf{y})$"
  2. every place "$Qx\phi(x, \mathbf{y})$" appears, replace it with "$\wedge_{\alpha<\omega_1} \phi(G^{0,\alpha}_\phi(\mathbf{y}), \mathbf{y})$"
  3. every place "$\neg Qx\phi(x, \mathbf{y})$" appears, replace it with "$\forall z(\phi(z, \mathbf{y}) \to \vee_{n<\omega} z = G^{1,n}_\phi(\mathbf{y})$"
- Let $\mathcal{F}_{Sk}$ be the smallest fragment of $\mathbb{L}_{\omega_2,\omega}(\tau)$ containing T_{Sk}.

Then $\mathcal{K}_{Sk} := (\text{Mod } T_{Sk}, \prec_{\mathcal{F}_{Sk}})$ is the Skolemization of $\mathcal{K}$ and the above results can be applied. Note that this Skolemization is $\mathbb{L}_{\infty,\omega}$-axiomatizable even though $\mathbb{L}(Q)$ is only $\mathbb{L}_{\omega_1,\omega_1}$-axiomatizable.

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