CATEGORICITY IN MULTIUNIVERSAL CLASSES

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Abstract. The third author has shown that Shelah's eventual categoricity conjecture holds in universal classes: class of structures closed under isomorphisms, substructures, and unions of chains. We extend this result to the framework of multiuniversal classes. Roughly speaking, these are classes with a closure operator that is essentially algebraic closure (instead of, in the universal case, being essentially definable closure). Along the way, we prove in particular that Galois (orbital) types in multiuniversal classes are determined by their finite restrictions, generalizing a result of the second author.

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1. Introduction

One of the most important test questions in the study of abstract elementary classes (AECs) is Shelah's eventual categoricity conjecture, i.e. the statement that if an AEC is categorical in any "sufficiently large" cardinal then it must be categorical on a tail of cardinals. This conjecture, if true, would be a generalization of Morley's theorem for first order theories to AECs.

An important class of AECs are the *universal classes*: classes of structures closed under isomorphism, substructures and union of $\subseteq$-increasing chains. Essentially by a result of Tarski, universal classes are precisely the classes of models of a universal sentence in $\mathcal{L}_{\infty,\omega}$ (see [Tar54]).

Date: April 24, 2018
AMS 2010 Subject Classification: Primary 03C48. Secondary: 03C45, 03C52, 03C55, 03C75.
Key words and phrases. Abstract elementary classes; Universal classes; Multiuniversal classes; Categoricity; Tameness.
This material is based upon work done while the first author was supported by the National Science Foundation under Grant No. DMS-1402191.

In a tour de force, the third author [Vas17a, Vas17b] proved Shelah's eventual categoricity conjecture for universal classes.

Fact 1.1 ([Vas17b, Theorem 7.3]). *Let* $\mathbf{K}$ *be a universal class. If* $\mathbf{K}$ *is categorical in some* $\lambda \geq \beth_{\beth_{\left(2^{|\tau(\mathbf{K})|+\aleph_0}\right)^+}}$, *then* $\mathbf{K}$ *is categorical in all* $\lambda \geq \beth_{\beth_{\left(2^{|\tau(\mathbf{K})|+\aleph_0}\right)^+}}$.

In this paper, we aim to generalize this result to a larger class of AECs which we call *multiuniversal*, see Definition 2.8. While there are many interesting universal classes (locally finite groups, valued fields with fixed value group, etc.), the restrictions on the definition of universal classes make it difficult to have universal classes that are both mathematically interesting and model-theoretically well-behaved. Recent work of Hyttinen and Kangas [HK] have given some confirmation of this by building on the third author's work to show that any universal class that is categorical in a high-enough successor must eventually look like either naked sets or vector spaces. The motivation for this generalization stems from the goal of slightly weakening the notion of universal class while still maintaining many of the nice mathematical properties of universal classes.

In aiming to generalize beyond universal classes, we took algebraically closed fields as our prototype. Fields are easily seen to be universal classes, but are model-theoretically intractable at that level of generality. Algebraically closed fields are the most model-theoretically well-behaved subclass of fields, but are not universal as they are not closed under subfield. One could make them into a universal class by expanding the language to include an n-ary function F for each n so that $F(a_0, \ldots, a_n)$ is a root of the polynomial $a_0 + \cdots + a_n x^n$. However, this choice of root function would destroy many of the nice model-theoretic properties of algebraically closed fields, e.g. the decision about whether the chosen fourth root of 2 squares to the chosen square root of 2 means that there are different complete theories in the expansion.

One solution to this dilemma would be to allow the addition of 'multifunctions' to the language, which are functions that are allowed to have a several output values (although the number of output values is fixed by the language). Then algebraically closed fields can be equipped with multifunctions F so that $F(a_0, \ldots, a_n)$ picks out the $n+1$-many roots of the polynomial $a_0 + \cdots + a_n x^n$. While this is good motivation, the explicit formulation of this framework is much more difficult as it requires specifying the syntax and semantics of these new multifunctions. Instead, we went with a definition of multiuniversal classes that more naturally fits in the standard model theoretic framework. First, we require that a multiuniversal class admits intersections (Definition 2.2). This is similar to having a notion of "submodel generated by," but doesn't require that the submodel is generated by terms in the language (as a universal class would). Instead, we want to think of the submodel as closing under imagined multifunctions. One consequence of this would be that, if x is in the closure of A, it is the image under some multifunction. Thus, the only other elements of the closure that have the same type over A are the other *finitely many* values of that multifunction. This motivates using the definition of a type being $\aleph_0$-algebraic (Definition 2.4). Although this seems weaker than an imagined multifunction framework, this definition of multiuniversal class allows us to prove nice structural results. Several examples beyond algebraically closed fields are listed in Example 2.9. A further alternate framework is to think of multiuniversal classes as universal classes that take value, not in the category of set, but in the category whose objects are sets X along with the collection of finite subsets of X.

The main result of the present paper is Corollary 4.3: Shelah's eventual categoricity conjecture holds for multiuniversal AECs. The proof goes along the same lines as the third author's proof of the corresponding result for universal classes [Vas17b]. An important step is to establish that any (not necessarily categorical) multiuniversal AEC has a strong locality property: its Galois (i.e. orbital) types are determined by their finite restrictions. In technical jargon, multiuniversal classes are fully $(< \aleph_0)$-tame and type-short, see Theorem 3.3. For universal classes, this is due to the second author and essentially follows from the fact that a partial isomorphism from A to B can be extended *uniquely* to the closure of A under the functions of the ambient model. In the setup of multiuniversal classes, there is no longer such a unique extension (think of the case of algebraically closed fields of characteristic zero: an automorphism of $\mathbb{Q}$ can be extended in two ways to $\mathbb{Q}(\sqrt{2})$). Still given any partial isomorphism p from A to B and a in the closure of A, there are only finitely-many extensions of p with domain $\mathrm{dom}(p) \cup \{a\}$. This will allow us to prove the result with a compactness argument (using Tychonoff's theorem). This can also be construed as a finite-injury proof.

2. Preliminaries

We assume the reader is familiar with the basics of AECs, as covered for example in Chapters 4 and 8 of [Bal09]. We will make heavy use of Galois (orbital) types and use the notation from [Vas16b, §2]. In particular, we let $\mathrm{gtp}(\bar{b}/A; N)$ denote the Galois type of the sequence $\bar{b}$ over A as computed inside N. We write $\mathbf{K}$ for an AEC, and write $\leq_{\mathbf{K}}$ for the ordering on $\mathbf{K}$. When $\mathbf{K}$ is clear from context, we may drop it from the name of a concept.

We begin by recalling some important properties an AEC may have. The following is from [Vas17a, Remark 4.12]:

Definition 2.1. *An AEC* $\mathbf{K}$ *has* weak amalgamation *if whenever* $gtp(a_1/M; N_1) = gtp(a_2/M; N_2)$*, there exists* $N_1' \leq_{\mathbf{K}} N_1$ *containing* a_1 *and* M *and there exists* $N \geq_{\mathbf{K}} N_2$ *and* $f : N_1' \xrightarrow[M]{} N$ *so that* $f(a_1) = a_2$*.*

The following was introduced for AECs by Baldwin and Shelah [BS08, Definition 1.2].

Definition 2.2. *Suppose that* $\mathbf{K}$ *is an AEC. For any* $A \subseteq N \in \mathbf{K}$ *define*

$$\mathrm{cl}^N_{\mathbf{K}}(A) := \bigcap \{N_0 \colon N_0 \leq_{\mathbf{K}} N \text{ and } A \subseteq |N_0|\}.$$

Note that we will omit $\mathbf{K}$ *when it is clear from context and we will not distinguish between the set* $\mathrm{cl}^N(A)$ *the the corresponding* $\tau(\mathbf{K})$*-structure induced from the* $\tau(\mathbf{K})$*-structure on* N*.*

We say that $\mathbf{K}$ *is an* AEC with intersections *(or* $\mathbf{K}$ has intersections*, or* $\mathbf{K}$ admits intersections*) if for all* $N \in \mathbf{K}$ *and* $A \subseteq |N|$*,* $\mathrm{cl}^N(A) \leq_{\mathbf{K}} N$*.*

We will heavily use the following basic properties of AECs with intersections [Vas17a, Propositions 2.14, 2.18, and Remark 4.13].

Fact 2.3. *Let* $\mathbf{K}$ *be an AEC with intersections.*

(1) *Let* $M \leq_{\mathbf{K}} N$ *and let* $A \subseteq |M|$. *Then* $\text{cl}^M(A) = \text{cl}^N(A)$.
(2) *(Finite character) Let* $M \in \mathbf{K}$ *and let* $a \in \text{cl}^M(B)$. *Then there exists a finite* $B_0 \subseteq B$ *such that* $a \in \text{cl}^M(B_0)$.
(3) *(Equality of types) Let* $M_1, M_2 \in \mathbf{K}$, $A \subseteq M_1 \cap M_2$. *Then*[1] $gtp(\bar{b}_1/A; M_1) = gtp(\bar{b}_2/A; M_2)$ *if and only if there exists an isomorphism* $f : \text{cl}^{M_1}(A\bar{b}_1) \to \text{cl}^{M_2}(A\bar{b}_2)$ *extending* f_0 *such that* $f(\bar{b}_1) = \bar{b}_2$.
(4) $\mathbf{K}$ *has weak amalgamation.*

The following essentially also appears in [She09b, Definition VI.1.15(2)].

Definition 2.4. *Let* μ *be a (possibly finite) cardinal. A type* $p \in gS^{<\infty}(A; N)$ *is* μ-algebraic *if for any* $N' \in \mathbf{K}$ *whose universe contains* A*, the set of realizations of* p *in* N' *has size strictly less than* μ.

The following basic properties of algebraicity will be useful:

Lemma 2.5. *Let* $\mathbf{K}$ *be an AEC with intersections and let* $p \in gS(A; N)$ *be a type realized in* $\text{cl}^N(A)$.

(1) *For any* $M \in \mathbf{K}$ *containing* A*, if* $\bar{b} \in M$ *realizes* p*, then* $\bar{b} \in \text{cl}^M(A)$.
(2) *For any* $M \in \mathbf{K}$*, if* p *is realized in* M*, then* p *has as many realizations in* M *as in* N.

Proof. Write $p = \text{gtp}(\bar{a}/A; N)$. For the first item, if $\bar{b}$ realizes p, then there is an isomorphism $f : \text{cl}^N(A\bar{a}) \cong_A \text{cl}^M(A\bar{b})$ such that $f(\bar{a}) = \bar{b}$. But $\text{cl}^N(A\bar{a}) = \text{cl}^N(A)$ by assumption, so $\text{cl}^M(A\bar{b}) = \text{cl}^M(A)$ and $\bar{b} \in \text{cl}^M(A)$.

For the second item, suppose that $\bar{b} \in M$ realizes p. Then $\bar{b} \in \text{cl}^M(A)$ by the first item and there exists an isomorphism $f : \text{cl}^N(A) \cong_A \text{cl}^M(A)$ sending $\bar{a}$ to $\bar{b}$. Without loss of generality, $N = \text{cl}^N(A)$ and $M = \text{cl}^M(A)$. Now as f is a bijection, it must send distinct realizations of p in N to distinct realizations of p in M and vice-versa. The result follows. □

Lemma 2.6. *Let* $\mathbf{K}$ *be an AEC with intersections. If* $p \in gS(A; N)$ *is* μ*-algebraic, then there exists* $\mu_0 < \mu$ *such that* p *is* μ_0^+*-algebraic.*

Proof. By Lemma 2.5. □

Lemma 2.7. *Let* $\mathbf{K}$ *be an AEC with intersections. Let* $M \in \mathbf{K}$ *and let* $A \subseteq |M|$. *Let* $\bar{b} \in {}^{\alpha}(\text{cl}^M(A))$. *Let* $p := gtp(\bar{b}/A; M)$. *Then* p *is* $(LS(\mathbf{K}) + \alpha)^+$*-algebraic.*

Proof. Pick $A_0 \subseteq A$ such that $|A_0| \leq \alpha + \aleph_0$ and $\bar{b} \in \text{cl}^M(A_0)$. Let $M_0 := \text{cl}^M(A_0)$. By Lemma 2.5, $p \upharpoonright A_0$ is $\|M_0\|^+$-algebraic. Now any element realizing p in M must realize $p \upharpoonright A_0$, so p is also $\|M_0\|^+$-algebraic. Since $\|M_0\| \leq \text{LS}(\mathbf{K}) + \alpha$, the result follows. □

Using algebraicity, we can give a semantic definition of multiuniversal classes:

[1]Note that by itself the statement "$A \subseteq M_1 \cap M_2$" is somewhat meaningless: the set A may be embedded in a different way inside M_1 and M_2. However both sides of the characterization of equality of types imply that A is embedded in the same way in M_1 and M_2.

Definition 2.8. *Let* $\mathbf{K}$ *be an AEC and let* μ *be a (possibly finite) cardinal. We say that* $\mathbf{K}$ *is* μ-multiuniversal *if it admits intersections and for any* $M \in \mathbf{K}$*, any* $A \subseteq |M|$*, and any* $b \in \text{cl}^M(A)$*,* $gtp(b/A; M)$ *is* μ*-algebraic. We call* $\mathbf{K}$ multiuniversal *if it is* $\aleph_0$*-multiuniversal.*

Note that 2-multiuniversal corresponds to universal (up to definable expansion of the vocabulary), see [LRV, Theorem 2.8]. Similarly, we could have defined precisely what is meant by a multifunction, and then defined a multiuniversal class to be a class of structures closed under isomorphisms, unions of increasing chains, and so that substructures are in a certain sense closed under multifunctions. We could then have shown that starting with a multiuniversal class in the sense of Definition 2.8, one adds a relation symbol for every Galois type of finite length, and then adds "Skolem multifunctions" to then get a multiuniversal class in the syntactic sense. We do not adopt this approach, since it is easier to work with multiuniversal classes semantically anyway.

Example 2.9.

(1) *Let* $\mathbf{K}$ *be an AEC with intersections. Then* $\mathbf{K}$ *is* $LS(\mathbf{K})^+$*-multiuniversal by Lemma 2.7.*
(2) *Any universal class is a multiuniversal AEC.*
(3) *Let* T *be a first-order theory with quantifier elimination. Let* $\mathbf{K}$ *be the class of all algebraically closed subsets of models of* T*, ordered by being a substructure. Then* $\mathbf{K}$ *is a multiuniversal AEC.*
(4) *Any AEC with intersections in which the closure operator is* locally finite *(i.e. the closure of any finite set is finite) is a multiuniversal AEC, see* [BKL17] *for a discussion of locally finite AECs.*
(5) *The AEC* $\mathbf{K}$ *of algebraically closed fields (ordered by subfield) is a multiuniversal AEC which is not a universal class.*
(6) *Let* K *be the class of pairs* (A, E)*, where* E *is an equivalence relation on* A*, each of whose classes are countably infinite. Order it by the relation "equivalence classes do not grow". The resulting AEC* $\mathbf{K}$ *is sometimes called the* toy quasiminimal class*. It is easy to check that* $\mathbf{K}$ *has intersections but is not* $\aleph_0$*-multiuniversal.*
(7) *Let* $\mathbf{K}$ *be the class of algebraically closed valued fields of rank one (that is, the valuation embeds into a subgroup of the reals). We code this by adding a constant symbol for every real number in the vocabulary. Then* $\mathbf{K}$ *is a multiuniversal AEC. Moreover,* $\mathbf{K}$ *is not axiomatizable by a first-order theory.*
(8) *Recall that a graph is* locally finite *if all its vertices have finite degree. Let* K *be the class of locally finite graphs, and make it into an AEC* $\mathbf{K}$ *by ordering it with* $G \leq_{\mathbf{K}} H$ *if and only if* G *is a subgraph of* H *and if* v *is in* H *and there is an edge from* v *to* G*, then* $v \in G$ *(that is, any connected component of* G *in* H *is* G*). Note that* $\mathbf{K}$ *admits intersections, is not first-order-axiomatizable, and is multiuniversal. Indeed, let* $M \in \mathbf{K}$*,* $A \subseteq |M|$*, and* $b \in \text{cl}^M(A)$*. Let* $A_0 \subseteq A$ *be finite such that* $b \in \text{cl}^M(A_0)$ *(Fact 2.3). It suffices to see that* $p := gtp(b/A_0; M)$ *is* $\aleph_0$*-algebraic. Note that* b *is connected to* A_0 *and the length* n *of the smallest path from* A_0 *to* b *is part of the information carried by the type of* b *over* A*. Now since* M *is locally*

finite, there are finitely-many vertices in M at distance at most n from A_0, and only those could realize p, so p is indeed $\aleph_0$-algebraic.

(9) *Let τ consist of unary predicates P and Q, of a binary relation E, and of binary relations $\langle R_m : m \in [2, \omega)\rangle$. We let K be the class of τ-structures M such that:*
   (a) *$P^M \cup Q^M = M$, $P^M \cap Q^M = \emptyset$, $E^M \subseteq P^M \times Q^M$. We think of Q^M as a set of subsets of P^M and of E^M as being the membership relation.*
   (b) *For each $s \in Q^M$, there exists at least one but only finitely-many $x \in P^M$ such that $xE^M s$. We let $n(s)$ be $|\{x \in P^M \mid xE^M s\}|$. Intuitively, Q^M consists of finite non-empty subsets of P^M, and $n(s)$ denotes the cardinality of the set coded by s.*
   (c) *If $xE^M s$ and $xE^M s'$, then $s = s'$. That is, there is at most one set containing each element.*
   (d) *For each $x \in P^M$, there exists $s \in Q^M$ such that $xE^M s$. That is, each element is contained in at least one set. This and the previous axioms imply that Q^M codes a partition of P^M consisting of finite sets.*
   (e) *For each $m \in [2, \omega)$, R_m is the graph of a bijection from $\{s \in Q^M \mid n(s) = 1\}$ onto $\{s \in Q^M \mid n(s) = m\}$. Thus R_m witnesses that in the partition there are as many sets with m elements as with one element.*

   We make K into an AEC $\mathbf{K}$ by ordering it by $M \leq_{\mathbf{K}} N$ if and only if $M \subseteq N$, and the sets do not grow: $xE^N s$ implies that $x \in N \backslash M$ if and only if $s \in N \backslash M$ (in particular the value of $n(s)$ is the same in N and M). Then $\mathbf{K}$ has intersections and $\mathbf{K}$ is a multiuniversal AEC (because, in a nutshell, $\text{cl}^M(A)$ adds only one set of each cardinality, and each such set has finitely-many elements). Moreover $\mathbf{K}$ is categorical in every infinite cardinal and is not first-order axiomatizable. Note also that $\mathbf{K}$ is not a universal class (because for a fixed s the elements x such that xEs all have the same type).

(10) *Let $\mathbf{K}$ be a universal class of abelian groups such that each group has finitely many n-torsion elements for every $n < \omega$ (and non-trivial n-torsion for at least one n). For example, the class of groups of the form $G \times \mathbb{Z}_n$, where G is abelian torsion-free and $\mathbb{Z}_n$ is the cyclic group of order n (coded by constant symbols in the language).*

   Let $\mathbf{K}^{div}$ be the class of consisting of the injective envelopes (generated divisible extensions) of groups in $\mathbf{K}$. Then $\mathbf{K}^{div}$ is not a universal class because there is no function that takes an element g to its divisors. However, the number of values for $\frac{g}{n}$ is precisely equal to the amount of n-torsion in the group. Thus, $\mathbf{K}^{div}$ is a multiuniversal class.

   If the original class $\mathbf{K}$ instead had n-torsion that was bounded above by μ, then the resulting class $\mathbf{K}$ would be μ^+-multiuniversal.

The following notation for certain threshold cardinals that come up often in the theory of AECs will be used:

Definition 2.10. *For a cardinal μ, we set $h(\mu) := \beth_{(2^\mu)^+}$. There will be a cardinal, $h(\mathbf{K})$ associated to an AEC which will be important. We refer the reader to* [Vas17b, Definition 2.16] *for a precise definition but note this cardinal satisfies $\beth_{LS(\mathbf{K})^+} \leq h(\mathbf{K}) \leq h(LS(\mathbf{K}))$.*

3. Some model theory of multiuniversal AECs

3.1. **Tameness and multiuniversal AECs.** It is natural to ask when a Galois type is determined by its restriction to small subsets. One can also ask when a type of a sequence is determined by its restriction to small subsequences. In the literature, the former property is called *tameness* [GV06] and the latter is called *type-shortness* [Bon14, Definition 3.3]. We amalgamate these two properties into one definition here:

Definition 3.1. *For κ an infinite cardinal, an AEC $\mathbf{K}$ is* $(<\kappa)$-short *if for any $N_1, N_2 \in \mathbf{K}$ and any ordinal α, if for $\ell = 1, 2$, $\bar{a}_\ell \in {}^\alpha N_\ell$ are such that $\text{gtp}(\bar{a}_1 \upharpoonright I/\emptyset; N_1) = \text{gtp}(\bar{a}_2 \upharpoonright I/\emptyset; N_2)$ for all $I \subseteq \alpha$ of size strictly less than κ, then $\text{gtp}(\bar{a}_1/\emptyset; N_1) = \text{gtp}(\bar{a}_2/\emptyset; N_2)$.*

Remark 3.2. *If $\mathbf{K}$ is $(<\kappa)$-short, then* [Bon14, Theorem 3.5] *$\mathbf{K}$ is fully $(<\kappa)$-tame and short over every set. That is, for any $N_1, N_2 \in \mathbf{K}$, any $A \subseteq |N_1| \cap |N_2|$, and any ordinal α, if for $\ell = 1, 2$, $\bar{a}_\ell \in {}^\alpha N_\ell$ are such that $\text{gtp}(\bar{a}_1 \upharpoonright I/A_0; N_1) = \text{gtp}(\bar{a}_2 \upharpoonright I/A_0; N_2)$ for all $I \subseteq \alpha$ and all $A_0 \subseteq A$ both of size strictly less than κ, then $\text{gtp}(\bar{a}_1/A; N_1) = \text{gtp}(\bar{a}_2/A; N_2)$.*

The second author has shown that, for any AEC $\mathbf{K}$, if $\kappa > \text{LS}(\mathbf{K})$ is strongly compact, then $\mathbf{K}$ is $(<\kappa)$-short [Bon14, Theorem 4.5]. Further, one can show in ZFC that universal classes are $(<\aleph_0)$-short due to the unique generating notion of closure under substructure (this appears as [Vas17a, Theorem 3.7] but is due to the second author). The notion of closure in multiuniversal classes is not as well behaved, but we show here that multiuniversal classes are still $(<\aleph_0)$-short. Here, however, the argument is much more complex: an element $b \in \text{cl}^M(A)$ cannot necessarily be written as a term of elements from A. Even if we moved to the formalism of "multifunction", this writing would not be unique. Instead, we use a sort of "finite injury" argument: given $\bar{a}_1$ and $\bar{a}_2$ that locally have the same type, we look at the space of partial mappings witnessing this. As we attempt to put the maps together, we might revise previous choices, but this revision happens only finitely many times at each point. This finiteness comes from the $\aleph_0$-algebraicity of types.

Theorem 3.3. *Any multiuniversal AEC is $(<\aleph_0)$-short.*

Proof. Let $N_1, N_2 \in \mathbf{K}$, let $\bar{a}_\ell \in {}^\alpha N_\ell$, $\ell = 1, 2$. Let $p_\ell := \text{gtp}(\bar{a}_\ell/\emptyset; N_\ell)$. Assume that $p_1^I = p_2^I$ for all finite $I \subseteq \alpha$ (here, p_ℓ^I denotes $\text{gtp}(\bar{a}_\ell \upharpoonright I/\emptyset; N_\ell)$). We have to show that $p_1 = p_2$. We will show that there is an isomorphism $f : \text{cl}^{N_1}(\bar{a}_1) \cong \text{cl}^{N_2}(\bar{a}_2)$ so that $f(\bar{a}_1) = \bar{a}_2$. Let $M_\ell := \text{cl}^{N_\ell}(\bar{a}_\ell)$. For $\ell = 1, 2$, let $A_\ell := \text{ran}(\bar{a}_\ell)$, and let $f_0 : A_1 \to A_2$ be the function sending $\bar{a}_1$ to $\bar{a}_2$.

Call a partial function f from $|M_1|$ to $|M_2|$ a (M_1, M_2)-*mapping* if for some (any) enumeration $\bar{b}$ of the domain of f, $\text{gtp}(\bar{b}/\emptyset; M_1) = \text{gtp}(f(\bar{b})/\emptyset; M_2)$. Set

$$P = \{B \subseteq |M_1| \mid B \text{ is finite and } B \subseteq \text{cl}^{M_1}(B \cap A_1)\}$$

For $B \in P$, there exists a (M_1, M_2)-mapping with domain B that agrees with f_0 on $A_1 \cap B$ (by the assumption that $p_1^I = p_2^I$ for all finite I, using Fact 2.3); let F_B to be the collection of all such mappings. For each $B \in P$, we claim that F_B is finite. To see this, we use the multiuniversality of $\mathbf{K}$: for each of the finitely many $b \in B \subseteq \text{cl}^{M_1}(B \cap A_1)$, $\text{gtp}(b/B \cap A_1; M_1)$ is $\aleph_0$-algebraic hence the set $\{f(b) \mid f \in F_B\}$ must be finite (note that $f[B \cap A_1] = f_0[B \cap A_1]$ for all $f \in F_B$).

Under the discrete topology, each F_B is compact since it is finite. Thus, by Tychonoff's Theorem, $F := \prod_{B \in P} F_B$ is compact when endowed with the product topology. Now for $B \in P$, write

$$P_B := \{\langle f_{B_0} \in F_{B_0} : B_0 \in P\rangle \mid \forall C \in \mathcal{P}(B) \cap P, f_C \subseteq f_B\}$$

$P_B \subseteq F$ is not empty, as any member of F_B induces an element of it. Moreover, it is a closed subset of F. Further, the collection $\{P_B : B \in P\}$ has the finite intersection property: for $B_0, B_1, \ldots, B_{n-1} \in P$, we have that $P_B \subseteq \bigcap_{i<n} P_{B_i}$ for any $B \in P$ extending $\bigcup_{i<n} B_i$. By the compactness of F, there is $\bar{f} = \langle f_B : B \in P\rangle \in \bigcap_{B \in P} P_B$.

We claim that $f := \bigcup_{B \in P} f_B$ is an isomorphism from M_1 to M_2 that sends $\bar{a}_1$ to $\bar{a}_2$. Each $b \in |M_1|$ is contained in some $B \in P$, so the domain of f is $|M_1|$. The coherence condition on P_B ensures that f is well-defined. These two conditions combined with the definition of F_B ensure that f extends f_0; that is, that f sends $\bar{a}_1$ to $\bar{a}_2$. Finally, f is surjective: let $b' \in M_2$. Fix $A_2' \subseteq A_2$ finite such that $b' \in \text{cl}^{M_2}(A_2')$. Let $A_1' := f_0^{-1}[A_2']$. Let $q' := \text{gtp}(b'/A_2'; M_2)$. We know that there exists $g : \text{cl}^{M_1}(A_1') \cong \text{cl}^{M_2}(A_2')$ such that g extends f_0. Let $b := g^{-1}(b')$, $q := \text{gtp}(b/A_1'; M_1)$. Let S' be the set of all realizations of q' in M_2. Note that S' is finite (Lemma 2.6) and contains b'. Let S be the set of all realizations of q in M_1. Note that $g[S] = S'$. But as g was arbitrary this implies that $f[S] = S'$. In particular, b is in the range of f, as desired. ☐

Remark 3.4. *Theorem 3.3 does* not *generalize to μ-multiuniversal classes: by Example 2.9(1), any AEC $\mathbf{K}$ with intersection is $LS(\mathbf{K})^+$-universal, but there are numerous examples of non-tame AECs with intersections (see e.g.* [BS08, BU17]*).*

3.2. **Abstract Morleyizations.** In [Vas16b, Definition 3.3], the third author introduced the *Galois Morleyization* of an AEC. It is an expansion of the vocabulary that adds predicates for each Galois types over the empty set (with length below a fixed bound). Following this, we say that an AEC $\mathbf{K}$ is $(< \aleph_0)$-*Morleyized* if for every $p \in \text{gS}^{<\omega}_{\mathbf{K}}(\emptyset)$, there is a relation $R_p \in \tau(\mathbf{K})$ (of arity $\ell(p)$) such that, for each $M \in \mathbf{K}$, $R_p(M) = p(M)$ (that is, R_p is realized exactly by the elements realizing p in M). By [Vas16b, Proposition 3.5], each AEC has a functorial expansion (see [Vas16b, Definition 3.1]) to a $(< \aleph_0)$-Morleyized AEC. By [Vas16b, Theorem 3.16], $\mathbf{K}$ is $(< \aleph_0)$-short if and only if Galois types are quantifier-free first-order types in the $(< \aleph_0)$-Galois Morleyization of $\mathbf{K}$. We give this conclusion a name:

Definition 3.5. *We say that an AEC $\mathbf{K}$* has quantifier-free types *if for any $N_1, N_2 \in \mathbf{K}$, $A \subseteq |N_1| \cap |N_2|$ and $\bar{a}_\ell \in {}^{<\infty}N_\ell$, $\ell = 1, 2$, we have $gtp(\bar{a}_1/A; N_1) = gtp(\bar{a}_2/A; N_2)$ if and only if $tp_{qf}(\bar{a}_1/A; N_1) = tp_{qf}(\bar{a}_2/A; N_2)$. Here, $tp_{qf}(\bar{a}/A; N)$ denotes the first-order quantifier-free type of $\bar{a}$ over A as computed in N.*

The discussion in the previous paragraph showed:

Fact 3.6. *Any $(< \aleph_0)$-short $(< \aleph_0)$-Morleyized AEC has quantifier-free types.*

We want to study the compactness behavior of types in AECs with quantifier-free types. The main result is Theorem 3.8, which gives a condition under which a strong form of finite satisfiability implies global satisfiability.

Definition 3.7. *Let* $\mathbf{K}$ *be an AEC and let* $p(\bar{x})$ *be a set of quantifier-free formulas in* $\tau(\mathbf{K})$*.*

(1) *We say that* p *is* $\mathbf{K}$-satisfiable *if there is* $M \in \mathbf{K}$ *and* $\bar{a} \in {}^{\ell(\bar{x})}M$ *such that* $M \models \phi[\bar{a}]$ *for all* $\phi \in p$*.*
(2) *We say that* p *is* strongly finitely $\mathbf{K}$-satisfiable *if for every finite subsequence* $\bar{x}_0$ *of* $\bar{x}$*,* $p \restriction \bar{x}_0$ *is* $\mathbf{K}$*-satisfiable, where* $p \restriction \bar{x}_0$ *denotes the set of formulas in* p *whose free variables are all in* $\bar{x}_0$*.*

Theorem 3.8 (Compactness theorem for AECs with quantifier-free types)**.** *Let* $\mathbf{K}$ *be an AEC with quantifier-free types and weak amalgamation (recall Definition 2.1). Let* $p(\bar{x})$ *be a complete set of quantifier-free formulas in* $\tau(\mathbf{K})$*. If* p *is strongly finitely* $\mathbf{K}$*-satisfiable, then* p *is* $\mathbf{K}$*-satisfiable.*

Note that the essential flavor of this proof is the argument that local AECs are compact, see [Bal09, Lemma 11.5].

Proof. We work by induction. Let $\alpha := \ell(\bar{x})$. Without loss of generality, α is an infinite cardinal and we have the result for all $\alpha_0 < \alpha$. For $I \subseteq \alpha$, write p^I for the restriction of p to formulas only using variables in $\{x_i \mid i \in I\}$. Recall that we are assuming that there is an equivalence between Galois types and quantifier-free types, and moreover we are also assuming that each p^I is complete.

Claim: For each $I \subseteq \alpha$ with $|I| < \alpha$, there is a unique Galois type $q_I \in \mathrm{gS}^I(\emptyset)$ such that for any $M \in \mathbf{K}$ and $\bar{b} \in {}^I M$, $\mathrm{gtp}(\bar{b}/\emptyset; M) = q_I$ if and only if $M \models \wedge p^I[\bar{b}]$.

Proof of Claim: When I is finite, this holds by completeness and strong finite consistency, using that every Galois type is represented by a formula. When I is infinite, use the induction hypothesis for existence and the equivalence between syntactic and Galois types for uniqueness. $\dagger_{\text{Claim}}$

Now we build $\langle M_i : i < \alpha \rangle$, $\langle f_{i,j} : i < j < \alpha \rangle$, and $\langle \bar{b}_i : i < \alpha \rangle$ such that for each $i \leq j \leq k < \alpha$:

(1) $M_i \in \mathbf{K}$, $f_{i,j} : M_i \to M_j$ is a $\mathbf{K}$-embedding, $\bar{b}_i \in {}^i M_i$, $f_{i,j}(\bar{b}_i) = \bar{b}_j \restriction i$.
(2) $f_{j,k} \circ f_{i,j} = f_{i,k}$, and $f_{i,i}$ is the identity.
(3) $\mathrm{gtp}(\bar{b}_i/\emptyset; M_i) = q_i$.

This is possible: we proceed inductively. When $i = 0$, this is easy. When $i = j+1$ is a successor, we use the uniqueness of q_j together with weak amalgamation. When i is limit, we take direct limits and use shortness.

This is enough: the direct limit of the system is as desired. ☐

Remark 3.9. *The prof shows that in Theorem 3.8, it is enough to assume that* $\mathbf{K}$ *is* $(< \aleph_0)$*-short only for types of length strictly less than* $\ell(p)$*.*

Corollary 3.10. *Let* $\mathbf{K}$ *be an AEC with quantifier-free types and intersections. Then any strongly finitely satisfiable complete set of quantifier-free formulas is satisfiable.*

Proof. By Fact 2.3, any AEC with intersections has weak amalgamation, so the result follows from Theorem 3.8. □

3.3. **Model-completeness.** Universal classes are examples of AECs whose **K**-substructure relation is just substructure. Following the first-order definition, Baldwin and Kolesnikov [BK09, Section 4] called these AECs *model-complete*.

Definition 3.11. *An AEC* $\mathbf{K}$ *is called* model-complete *if for any* $M, N \in \mathbf{K}$, $M \leq_{\mathbf{K}} N$ *holds if and only if* $M \subseteq N$.

Not all model-complete AECs are universal classes (for example, algebraically closed fields are not universal). In fact, any time that the AEC is finitary (in the sense of Hyttinen and Kesälä [HK06]), we can expand the vocabulary to obtain a model-complete AEC [BV, Theorem 3.14]. We will use the following special case in this paper:

Lemma 3.12. *Any AEC with quantifier-free types is model-complete.*

Proof. Let $\mathbf{K}$ be an AEC with quantifier-free types. Let $M, N \in \mathbf{K}$ be such that $M \subseteq N$. Let $\bar{a}$ be an enumeration of M. It is enough to show that $\mathrm{gtp}(\bar{a}/\emptyset; M) = \mathrm{gtp}(\bar{a}/\emptyset; N)$ (then the definition of Galois types gives that $M \leq_{\mathbf{K}} N$). But this holds because $M \subseteq N$ and quantifier-free types are the same as Galois types. □

3.4. **Isolation.** An interesting feature of multiuniversal classes is that types are isolated over finite subtypes:

Definition 3.13. *Let* $\mathbf{K}$ *be an AEC and let* $M \in \mathbf{K}$. *Let* $A, B \subseteq |M|$. *Let* $p \in gS(A; M)$, $q \in gS(B; M)$. *We say that* p isolates q in M *if whenever* $p = gtp(\bar{b}/A; M)$, *we have that* $q = gtp(\bar{b}/B; M)$.

Definition 3.14. *An AEC* $\mathbf{K}$ *with intersections satisfies the* isolation axiom *if whenever* $M \in \mathbf{K}$, $A \subseteq |M|$, *and* $\bar{b} \in {}^{<\omega}M$ *is so that* $\bar{b} \in \mathrm{cl}^M(A)$, *then there exists* $A_0 \subseteq A$ *finite such that* $gtp(\bar{b}/A_0; M)$ *isolates* $gtp(\bar{b}/A; M)$ *in* M.

Theorem 3.15. *Any multiuniversal class satisfies the isolation axiom.*

Proof. Let $\mathbf{K}$ be a multiuniversal class. Let $M \in \mathbf{K}$, $A \subseteq |M|$, and $\bar{b} \in {}^{<\omega}M$ be such that $\bar{b} \in \mathrm{cl}^M(A)$. Let $p := \mathrm{gtp}(\bar{b}/A; M)$. Fix $A_0 \subseteq A$ finite such that $\bar{b} \in \mathrm{cl}^M(A_0)$. By definition, $p \upharpoonright A_0$ is $\aleph_0$-algebraic, hence by Lemma 2.6 is $(n+1)$-algebraic for some $n < \omega$. Let $\bar{b}_0, \ldots, \bar{b}_{n-1}$ be all the realizations of $p \upharpoonright A_0$ inside M. Now pick $A_1 \subseteq A$ finite such that $A_0 \subseteq A_1$ and for any $i < j < n$, $\mathrm{gtp}(\bar{b}_i/A; M) \neq \mathrm{gtp}(\bar{b}_j/A; M)$ implies that $\mathrm{gtp}(\bar{b}_i/A_1; M) \neq \mathrm{gtp}(\bar{b}_j/A_1; M)$. This is possible by tameness (Theorem 3.3 and Remark 3.2). Now it is easy to check that $p \upharpoonright A_1$ isolates p in M. □

3.5. **A topological characterization of multiuniversal classes.** While this is not needed for the rest of the paper, it is interesting to note that multiuniversal classes can be characterized in terms of the compactness of certain automorphism groups (we say that a group G of automorphism of a structure M_0 is *compact* if it is compact in the product space ${}^{M_0}M_0$, where M_0 itself is given the discrete topology).

Theorem 3.16. *Let* $\mathbf{K}$ *be an AEC with intersections. The following are equivalent:*

(1) $\mathbf{K}$ *is multiuniversal.*
(2) *For any* $M \in \mathbf{K}$ *and any* $A \subseteq |M|$, $\mathrm{Aut}_A(\mathrm{cl}^M(A))$ *is compact.*
(3) *For any* $M \in \mathbf{K}$ *and any* finite $A \subseteq |M|$, $\mathrm{Aut}_A(\mathrm{cl}^M(A))$ *is compact.*

Proof. It is folklore (see for example Section 4.1 of [Mac11]) that for any structure M_0 and any subset $A \subseteq |M_0|$, the group G of automorphisms of M_0 fixing A is compact if and only if for any $\bar{b} \in {}^{<\omega}M_0$, the orbit of $\bar{b}$ under the action of G on M_0 is finite. We will use this throughout the proof.

- (1) implies (2): if $\mathbf{K}$ is multiuniversal, $M \in \mathbf{K}$, $A \subseteq |M|$, $M_0 := \mathrm{cl}^M(A)$, and $G := \mathrm{Aut}_A(M_0)$, then for any $\bar{b} \in {}^{<\omega}M_0$, the orbit of $\bar{b}$ under the action of G on M_0 must be finite because $\mathrm{gtp}(\bar{b}/A; M)$ is $\aleph_0$-algebraic by assumption. Thus G is compact.
- (2) implies (3): trivial.
- (3) implies (1): Assume (3) and let $M \in \mathbf{K}$, $A \subseteq |M|$. Let $M_0 := \mathrm{cl}^M(A)$. We need to check that any type in $\mathrm{gS}(A; M_0)$ is $\aleph_0$-algebraic. So let $p = \mathrm{gtp}(b/A; M_0) \in \mathrm{gS}(A; M_0)$. Fix a finite $A_0 \subseteq A$ such that $b \in \mathrm{cl}^M(A_0)$ (Fact 2.3). It suffices to see that $p_0 := \mathrm{gtp}(b/A_0; M)$ is $\aleph_0$-algebraic. Let $M_0' := \mathrm{cl}^M(A_0)$. By Lemma 2.5, M_0' must contain all the realizations of p_0. Moreover, for any $b' \in M_0'$ of p_0, there must exist an automorphism of M_0' fixing A_0 sending b' to b (by Fact 2.3). Since $G := \mathrm{Aut}_{A_0}(M_0')$ is compact by assumption, the orbit of b under the action of G must be finite, hence p_0 is $\aleph_0$-algebraic, as desired.

□

4. Eventual categoricity

We now come to the main result of the paper. The goal of this section is to prove the following (recall that $h(\mathbf{K})$ was defined in Definition 2.10).

Theorem 4.1. *Let* $\mathbf{K}$ *be an AEC. Assume that:*

(1) $\mathbf{K}$ *has intersections (Definition 2.2).*
(2) $\mathbf{K}$ *has quantifier-free types (Definition 3.5).*
(3) $\mathbf{K}$ *satisfies the isolation axiom (Definition 3.14).*

If $\mathbf{K}$ *is categorical in* some $\lambda \geq \beth_{h(\mathbf{K})}$, *then* $\mathbf{K}$ *is categorical in* all $\lambda' \geq \beth_{h(\mathbf{K})}$.

Note that universal classes are AECs with quantifier-free types and intersections satisfying the isolation axiom (intersections hold by [Vas17a, Example 2.6(1)], having quantifier-free types holds by [Vas17a, Remark 3.8], and the isolation axiom holds by Theorem 3.15). Thus Theorem 4.1 generalizes the categoricity theorem for universal classes of the third author [Vas17b, Theorem 7.3]. In fact the proof is essentially the same, but one has to check that everything still goes through.

It is also worth noting that while multiuniversal classes satisfy the conditions of Theorem 4.1, there are AECs which satisfy these conditions but are not multiuniversal. In particular Example 2.9(6) is one such AEC.

Theorem 4.1 also gives us our desired categoricity transfer for multiuniversal classes as a corollary of our previous analysis.

Corollary 4.2. *Let* $\mathbf{K}$ *be a* $(< \aleph_0)$*-short AEC with intersections which satisfies the isolation axiom. If* $\mathbf{K}$ *is categorical in* some $\lambda \geq \beth_{h(2^{LS(\mathbf{K})})}$*, then* $\mathbf{K}$ *is categorical in* all $\lambda' \geq \beth_{h(2^{LS(\mathbf{K})})}$.

Proof. Morleyize by adding at most $2^{\mathrm{LS}(\mathbf{K})}$-relation symbols for Galois types of finite length (see [Vas16b, §3]). We obtain an AEC with Löwenheim-Skolem-Tarski number at most $2^{\mathrm{LS}(\mathbf{K})}$ which satisfies the hypotheses of Theorem 4.1 and the Moleyization preserves categoricity. □

Corollary 4.3. *Let* $\mathbf{K}$ *be a multiuniversal AEC. If* $\mathbf{K}$ *is categorical in* some $\lambda \geq \beth_{h(2^{LS(\mathbf{K})})}$*, then* $\mathbf{K}$ *is categorical in* all $\lambda' \geq \beth_{h(2^{LS(\mathbf{K})})}$.

Proof. By definition, $\mathbf{K}$ has intersections. The other two conditions of Corollary 4.2 are given by Theorems 3.3 and 3.15. □

The proof of Theorem 4.1 goes along the lines of the corresponding result for universal classes proven in [Vas17b, Theorem 7.3]. We will quote and use terminology freely from [Vas17b] (particularly Section 6) and [She09b, Section V.B]. The reader is advised to have copies open on their desk as they read this proof, although we will try to give a sense of the definitions used. The proof traces back to checking that some arguments of Shelah from [She87] (we will use the revised version from [She09b, Chapter V]) still go through in the setup of this paper.

A crucial tool in this work is the use of averages. Recall that we have moved to a context where Galois types are quantifier-free. This allows us to make the following definition, following [She09b, Section V.A.2]: given $\{\bar{b}_i \mid i \in I\}, A \subseteq M$ and a cardinal χ,

$$\begin{aligned}\mathrm{Av}_\chi\left(\{\bar{a}_i \mid i \in I\}/A; M\right) := \{\phi(\bar{x}, \bar{a}) \quad | \quad & \phi \text{ is quantifier free and} \\ & \text{for all but } < \chi\text{-many } i,\ M \vDash \phi(\bar{b}_i, \bar{a})\}\end{aligned}$$

Not every sequence will give rise to a complete average, but we call sequences that do *convergent* (after suppressing some parameters). The main technique for finding convergent sequences is [She09b, Theorem V.A.2.8], which says that, if the class fails to have an appropriate order property, then any sufficiently large set of parameters can be 'pruned' to a convergent subset of the same size. The assumption of no order property follows from categoricity by [Vas17b, Lemma 7.1].

It suffices to prove [Vas17b, Fact 6.10] for classes satisfying the assumptions of Theorem 4.1. Then the rest of the proof of [Vas17b, Theorem 7.3] is exactly the same. More precisely, we will show:

Theorem 4.4. *Let* $\mathbf{K}$ *be an AEC. Assume that:*

1. $\mathbf{K}$ *has intersections.*
2. $\mathbf{K}$ *has quantifier-free types.*
3. $\mathbf{K}$ *satisfies the isolation axiom.*
4. $\chi \geq LS(\mathbf{K})$ *is such that* $\mathbf{K}$ *does not have the order property of length* χ^+.

Set $\mu := 2^{2^\chi}$ and let $\mathbf{K}^0 := (K, \leq^{\chi^+,\mu^+})$ ($\leq^{\chi^+,\mu^+}$ is the ordering defined in [She09b, Definition V.A.4.1]*: roughly it requires that $M \subseteq N$, and the type of any $\bar{b} \in {}^{<\omega}N$ over M is the average of some sequence from M). Then:*

(1) *K^0 is a weak AEC with $LS(\mathbf{K}^0) \leq \mu^+$ (this means it satisfies the axioms of an AEC except perhaps smoothness of unions).*
(2) *Let cl^M be the closure operator on $\mathbf{K}$ and let $\mathop{\smile}\limits^{\mid}$ be the 4-ary relation defined in the statement of* [Vas17b, Fact 6.10]*; the key condition is that $M_1 \underset{M_0}{\overset{M_3}{\mathop{\smile}\limits^{\mid}}} M_2$ implies that the Galois type of any $\bar{a} \in M_1$ over M_2 is the average of a sequence from M_0.*
We then have that $(\mathbf{K}^0, \mathop{\smile}\limits^{\mid}, \mathrm{cl})$ satisfies $AxFr_1$ (see [She09b, Definition V.B.1.6]*). Moreover,*
   (a) *cl is algebraic (see* [Vas17b, Definition 5.22]*); and*
   (b) *$\mathop{\smile}\limits^{\mid}$ is μ^+-based (see* [Vas17b, Definition 4.12]*).*

Proof of Theorem 4.1. By the proof of [Vas17b, Theorem 7.3], substituting Theorem 4.4 to [Vas17b, Fact 6.10]. □

Proof of Theorem 4.4. That $\mathbf{K}^0$ is a weak AEC with $\mathrm{LS}(\mathbf{K}^0) \leq \mu^+$ is as in the proof of [She09b, Lemma V.B.2.9]. Now cl is algebraic because $\mathbf{K}$ is model-complete (Fact 3.12). That $\mathop{\smile}\limits^{\mid}$ is μ^+-based is proven exactly as in the proof of [Vas17b, Fact 6.10]. It remains to see that $(\mathbf{K}^0, \mathop{\smile}\limits^{\mid}, \mathrm{cl})$ satisfies AxFr_1. From now on, we write "convergent" instead of "(χ^+, μ^+)-convergent", "averageable" instead of "(χ^+, μ^+)-averageable", and "$\mathrm{Av}(\mathbf{I}/A; M)$" instead of "$\mathrm{Av}_{\chi^+}(\mathbf{I}/A; M)$".

For this we go through Shelah's proof in [She09b, Lemma V.B.2.9]. A key claim will be:

Claim: Let $M_0, M \in K$ with $M_0 \subseteq M$. Let $\bar{a}, \bar{b} \in {}^{<\omega}M$ be such that $\bar{b} \in \mathrm{cl}^M(\bar{a})$ and $\mathrm{gtp}(\bar{b}/\bar{a}; M)$ isolates $\mathrm{gtp}(\bar{b}/M_0; M)$ in M. If $\mathrm{tp}_{\mathrm{qf}}(\bar{a}/M_0; M)$ is averageable over M_0 in M, then so is $\mathrm{tp}_{\mathrm{qf}}(\bar{b}/M_0; M)$.

Proof of Claim: Note that Galois and quantifier-free types are interchangeable by hypothesis. Let $\mathbf{I} := \langle \bar{a}_i : i \in I \rangle$ be a sequence of elements in M_0 of arity $\ell(\bar{a})$ which is convergent such that $\mathrm{Av}(\mathbf{I}/M_0; M) = \mathrm{tp}_{\mathrm{qf}}(\bar{a}/M_0; M)$. Since $\mathbf{I}$ has at least μ^+-many elements, we can prune it further to assume without loss of generality that $\mathrm{tp}_{\mathrm{qf}}(\bar{a}_i/\emptyset; M) = \mathrm{tp}_{\mathrm{qf}}(\bar{a}/\emptyset; M)$ for all $i \in I$. Thus for each $i \in I$ there exists $f_i : \mathrm{cl}^{M_0}(\bar{a}_i) \cong \mathrm{cl}^M(\bar{a})$ sending $\bar{a}_i$ to $\bar{a}$. For $i \in I$, let $\bar{b}_i := f_i^{-1}(\bar{b})$. Let $\mathbf{J} := \langle \bar{b}_i : i \in I \rangle$. By pruning, we can assume without loss of generality that $\mathbf{J}$ is convergent. We want to see that $\mathrm{Av}(\mathbf{J}/M_0; M) = \mathrm{tp}_{\mathrm{qf}}(\bar{b}/M_0; M)$.

Let $\phi(\bar{x}; \bar{c}) \in \mathrm{Av}(\mathbf{J}/M_0; M)$. This means that for all but at most χ-many $i \in I$, $M \models \phi[\bar{b}_i; \bar{c}]$. Let $p := \mathrm{tp}_{\mathrm{qf}}(\bar{a}\bar{b}/\emptyset; M)$. By construction of $\bar{b}_i$, $p = \mathrm{tp}_{\mathrm{qf}}(\bar{a}_i\bar{b}_i/\emptyset; M)$ for all $i \in I$. Now by averageability, for most $i \in I$, $\mathrm{tp}_{\mathrm{qf}}(\bar{a}_i/\bar{c}; M) = \mathrm{tp}_{\mathrm{qf}}(\bar{a}/\bar{c}; M)$. Let $g_i : \mathrm{cl}^M(\bar{a}_i) \cong_{\bar{c}} \mathrm{cl}^M(\bar{a})$ send $\bar{a}_i$ to $\bar{a}$, and let $\bar{b}' := f(\bar{b}_i)$. Then for all such i, $M \models \phi[\bar{b}'; \bar{c}] \wedge p[\bar{b}'; \bar{a}]$. Thus $\mathrm{gtp}(\bar{b}/\bar{a}; M) = \mathrm{gtp}(\bar{b}'/\bar{a}; M)$, but by isolation this means that $\mathrm{gtp}(\bar{b}'/M_0; M) = \mathrm{gtp}(\bar{b}/M_0; M)$, so in particular $M \models \phi[\bar{b}; \bar{c}]$, as desired. $\dagger_{\text{Claim}}$

We now prove all the axioms from [She09b, Section V.B.§1]. Clearly, $\mathop{\smile\!\!\!\!\!\!|}$ and cl are preserved under isomorphisms and the axioms from group A hold. By Fact 2.3, the finite character axiom C.7 holds, as do the axioms from group B. Thus as in the proof of [She09b, Lemma V.B.2.9], it suffices to show [She09b, Sublemmas V.B.2.10-2.13].

(1) Existence (axiom C2): Let $M_0 \leq_{\mathbf{K}^0} M_\ell$, $\ell = 1, 2$. First, we build $M_3 \in \mathbf{K}^0$ and $\mathbf{K}$-embeddings $f_\ell : M_\ell \to M_3$ for $\ell = 1, 2$ such that
   (a) $f_1 \restriction M_0 = f_2 \restriction M_0$;
   (b) $f_1[M_1] \underset{f_1[M_0]}{\overset{M_3}{\mathop{\smile\!\!\!\!\!\!|}}} f_2[M_2]$; and
   (c) $M_3 = \mathrm{cl}_3^M(f_1[M_1] \cup f_2[M_2])$.

   Note that we do not claim that the f_ℓ are $\mathbf{K}^0$-embeddings, although this will follow from (4) and Symmetry below.

   Let $\lambda := \|M_1\| + \|M_2\| + \aleph_0$. For $\ell = 1, 2$, let $\bar{c}^\ell := \langle c_i^\ell : i < \lambda \rangle$ be an enumeration (possibly with repetitions) of M_ℓ. For $u \subseteq \lambda$, write $\bar{c}_u^\ell$ for $\bar{c}^\ell \restriction u$. For each finite $u \subseteq \lambda$, by definition of $\leq_{\mathbf{K}^0}$, there is a convergent sequence $\mathbf{I}_u$ inside M_0 such that $\mathrm{tp}_{\mathrm{qf}}(\bar{c}_u^1/M_0; M_1) = \mathrm{Av}(\mathbf{I}_u/M_0; M_1)$. Let $q_u := \mathrm{Av}(\mathbf{I}_u/M_2; M_2)$, seen as a type in the variables $\bar{x}_u^1 := \langle x_i^1 : i \in u \rangle$. Now let p_u be the set of quantifier-free formulas $\phi(\bar{x}_u^1; \bar{x}_u^2)$ such that $\phi(\bar{x}_u^1; \bar{c}_u^2) \in q_u$. Let $p := \bigcup_{u \in [\lambda]^{<\aleph_0}} p_u$. Note that p is complete as a quantifier-free type.

   Now, for each finite $u \subseteq \lambda$, p_u contains at most $(|\tau(\mathbf{K})| + \aleph_0)$-many formulas and $\mathbf{I}_u$ has the much bigger size μ^+. Moreover for each $\phi \in p_u$, all but fewer than μ^+-many elements of $\mathbf{I}_u$ satisfy $\phi(\bar{x}_u^1; \bar{c}_u^2)$. It follows that p_u is realized in M_2. Thus p is strongly finitely $\mathbf{K}$-satisfiable. By Corollary 3.10, p is $\mathbf{K}$-satisfiable. Let $\bar{d}^1\bar{d}^2$ realize p (where $\bar{d}^\ell$ realize $p \restriction \bar{x}^\ell$) inside some $M \in \mathbf{K}$. Let $M_3 := \mathrm{cl}^M(\bar{d}^1\bar{d}^2)$. Now the formula "$\bar{x}_u^1 = \bar{x}_v^2$" is in p whenever $\bar{c}_u^1 = \bar{c}_v^2$ are in M_0. Moreover, $\mathrm{tp}_{\mathrm{qf}}(\bar{d}^\ell/\emptyset; M_3) = \mathrm{tp}_{\mathrm{qf}}(\bar{c}^\ell/\emptyset; M_1)$. Since Galois and quantifier-free types are the same (in $\mathbf{K}$), sending $\bar{c}^\ell$ to $\bar{d}^\ell$ is a $\mathbf{K}$-embedding $f_\ell : M_\ell \to M_3$ and $f_1 \restriction M_0 = f_2 \restriction M_0$. By construction, $f_1[M_1] \underset{f_1[M_0]}{\overset{M_3}{\mathop{\smile\!\!\!\!\!\!|}}} f_2[M_2]$.

(2) Uniqueness (axiom C5): By [She09b, Claim V.A.4.6(2)], the fact that quantifier-free types are the same as Galois types, and the argument in [Vas16a, Lemma 12.6].

(3) Symmetry (axiom C6): Exactly as in [She09b, Sublemma V.B.2.11].

(4) If $M_1 \underset{M_0}{\overset{M_3}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$, then $M_2 \leq_{\mathbf{K}^0} M_3$ ([She09b, Claim V.B.2.12]): By definition of $\mathop{\smile\!\!\!\!\!\!|}$ and transitivity of $\leq_{\mathbf{K}^0}$ we can assume without loss of generality that $M_3 = \mathrm{cl}^{M_3}(M_1 \cup M_2)$. Let $\bar{b} \in {}^{<\omega}M_3$. We have to see that $\mathrm{tp}_{\mathrm{qf}}(\bar{b}/M_2; M_3)$ is averageable over M_2. By the isolation axiom, there is a finite $A \subseteq |M_1| \cup |M_2|$ such that $\mathrm{tp}_{\mathrm{qf}}(\bar{b}/M_2; M_3)$ is isolated by $\mathrm{tp}_{\mathrm{qf}}(\bar{b}/A; M_3)$ in M_3. Let $\bar{a}$ be an enumeration of A. By the Claim, it suffices to see that $\mathrm{tp}_{\mathrm{qf}}(\bar{a}/M_2; M_3)$ is averageable over M_2 in M_3. Now by definition of $\mathop{\smile\!\!\!\!\!\!|}$, $\mathrm{tp}_{\mathrm{qf}}(\bar{a}/M_2; M_3)$ is averageable over M_0, hence over M_2, as desired.

(5) Base enlargement (axiom C4): Assume $M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M_2$ and $M_0 \leq_{\mathbf{K}^0} M_2' \leq_{\mathbf{K}^0} M_2$. We want to see that $\mathrm{cl}^{M_3}(M_2' \cup M_1) \underset{M_2'}{\overset{M_3}{\downarrow}} M_2$. Now by monotonicity we know that $M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M_2'$, hence by definition of $\downarrow$ and the previous part $M_2' \leq_{\mathbf{K}^0} \mathrm{cl}^{M_3}(M_2' \cup M_1)$. Also, $\mathrm{cl}^{M_3}(\mathrm{cl}^{M_3}(M_2' \cup M_1) \cup M_2) = \mathrm{cl}^{M_3}(M_1 \cup M_2) \leq_{\mathbf{K}^0} M_3$ by definition of $\downarrow$ and the assumption that $M_1 \underset{M_0}{\overset{M_3}{\downarrow}} M_2$. It remains to see that for any $\bar{c} \in {}^{<\omega}\,\mathrm{cl}^{M_2' \cup M_1}$, $\mathrm{tp}_{\mathrm{qf}}(\bar{c}/M_2; M_3)$ is averageable over M_2'.

First we show the transitivity property of $\downarrow$: if $N_1 \underset{N_0}{\overset{N_3}{\downarrow}} N_2$ and $N_3 \underset{N_2}{\overset{N_5}{\downarrow}} N_4$, then $N_1 \underset{N_0}{\overset{N_5}{\downarrow}} N_4$. To see this, let $\bar{b} \in {}^{<\omega}N_1$. We want to see that $\mathrm{tp}(\bar{b}/N_4; N_5)$ is averageable over N_0, and we know that $\mathrm{tp}(\bar{b}/N_4; N_5)$ is averageable over N_2 and $\mathrm{tp}(\bar{b}/N_2; N_5)$ is averageable over N_0. To conclude what we want, imitate the proof of [She09a, Claim II.2.18], noting that base monotonicity is trivial for the notion of being averageable over.

Now that we have transitivity, we can conclude base enlargement on general grounds: as in the proof of [She09a, Claim III.9.6(E)(b)], there is M_1' and M_3' such that $M_3 \leq_{\mathbf{K}^0} M_3'$, $M_1 \leq_{\mathbf{K}} M_1'$, and $M_1' \underset{M_2'}{\overset{M_3'}{\downarrow}} M_2$. Now observe that $\mathrm{cl}^{M_3'}(M_1 \cup M_2') = \mathrm{cl}^{M_3}(M_1 \cup M_2')$ to conclude.

□

References

[Bal09] John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, 2009.

[BK09] John T. Baldwin and Alexei Kolesnikov, *Categoricity, amalgamation, and tameness*, Israel Journal of Mathematics **170** (2009), 411–443.

[BKL17] John T. Baldwin, Martin Koerwien, and Michael C. Laskowski, *Disjoint amalgamation in locally finite AEC*, The Journal of Symbolic Logic **82** (2017), no. 1, 98–119.

[Bon14] Will Boney, *Tameness from large cardinal axioms*, The Journal of Symbolic Logic **79** (2014), no. 4, 1092–1119.

[BS08] John T. Baldwin and Saharon Shelah, *Examples of non-locality*, The Journal of Symbolic Logic **73** (2008), 765–782.

[BU17] Will Boney and Spencer Unger, *Large cardinal axioms from tameness in AECs*, Proceedings of the American Mathematical Society **145** (2017), 4517–4532.

[BV] Will Boney and Sebastien Vasey, *Structural logic and abstract elementary classes with intersections*, Preprint. URL: `http://arxiv.org/abs/1801.01908v2`.

[GV06] Rami Grossberg and Monica VanDieren, *Galois-stability for tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.

[HK] Tapani Hyttinen and Kaisa Kangas, *Categoricity and universal classes*, Preprint. URL: `https://arxiv.org/abs/1712.08532v2`.

[HK06] Tapani Hyttinen and Meeri Kesälä, *Independence in finitary abstract elementary classes*, Annals of Pure and Applied Logic **143** (2006), 103–138.

[LRV] Michael J. Lieberman, Jiří Rosický, and Sebastien Vasey, *Universal abstract elementary classes and locally multipresentable categories*, Preprint. URL: `http://arxiv.org/abs/1707.09005v3`.

[Mac11] Dugald Macpherson, *A survey of homogeneous structures*, Discrete Mathematics **311** (2011), no. 15, 1599–1634.

[She87] Saharon Shelah, *Universal classes*, Classification theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 264–418.

[She09a] ———, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[She09b] ———, *Classification theory for abstract elementary classes 2*, Studies in Logic: Mathematical logic and foundations, vol. 20, College Publications, 2009.

[Tar54] Alfred Tarski, *Contributions to the theory of models I*, Indagationes Mathematicae **16** (1954), 572–581.

[Vas16a] Sebastien Vasey, *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 11, 1029–1092.

[Vas16b] ———, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), 567–592.

[Vas17a] ———, *Shelah's eventual categoricity conjecture in universal classes: part I*, Annals of Pure and Applied Logic **168** (2017), no. 9, 1609–1642.

[Vas17b] ———, *Shelah's eventual categoricity conjecture in universal classes: part II*, Selecta Mathematica **23** (2017), no. 2, 1469–1506.

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