CLASSIFICATION THEORY FOR TAME ABSTRACT ELEMENTARY CLASSES
MATH 255 LECTURE NOTES

WILL BONEY

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These notes are lecture notes for Math 255 "Classification Theory for Tame Abstract Elementary Classes." Some good general references are [Gro02, BV17b, Bal09, She09b, Gro1X]. See the course website [Bonb] or syllabus for a discussion.

1. Shopping Day

The goal of today is to explain why you might want to take this course. Slides for a more in-depth but similar argument (that is a few years old) can be found on my website [Bone].

1.1. **Classification theory for Elementary Classes.**
Here's a nice introduction to classification theory: consider vector space over $\mathbb{R}$ or some other fixed field K. From linear algebra, we know the following:

- Each vector space has a basis, which is a spanning and linearly independent set.
- Every basis in a vector space has the same size, which we can call the dimension d.

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- This dimension characterizes the vector fields up to isomorphism in the following sense: if V_1 and V_2 are vector fields with the same dimension, then $V_1 \cong V_2$

Note that we get the following corollary: if V_1 and V_2 are vector fields of the same size and are larger than $|\mathbb{R}|$, then $V_1 \cong V_2$. This is because the dimension plus size of the base field gives the size of the vector space.

Now consider algebraically closed fields of characteristic 0 (ACF_0's). From Galois theory, we might know the following:

- Each ACF_0 K has a *transcendence basis*, which is a set of elements X that are all mutually transcendental over the base field $\mathbb{Q}$ and every element of K is a root of a polynomial with coefficients from $\mathbb{Q}$ and X.
- Every transcendence basis has the same size, which we call the transcendence degree t.
- This dimension characterizes the ACF_0 up to isomorphism in the same sense as above: if K_1 and K_2 are ACF_0's with the same transcendence degree, then $K_1 \cong K_2$.

Note that we get the following corollary: if K_1 and K_2 are ACF_0 of the same uncountable size, then $K_1 \cong K_2$.

These are both instances[1] of *Morley's Theorem*, which is often cited as the birth of modern model theory (proved by Michael Morley in [Mor65a] and later improved by Saharon Shelah in [She74]).

Theorem 1.1 (Morley's Theorem)**.** *Suppose that T is a countable first-order theory. If there is exactly one model up to isomorphism in* some *uncountable cardinal, then there is exactly one model up to isomorphism in* every *uncountable cardinal.*

Proof sketch for countable theories (due to John Baldwin and Alistair Lachlan [BL71]): Assume that there is exactly one model up to isomorphism in some uncountable cardinal.

- Given a structure M modeling T, we can find a nice subset X of M that controls M (in the sense that it is enough to find an isomorphism for the nice subsets). In these examples, the nice subset is the whole set.
- On these nice subsets, we can define a closure relation cl that acts *span* in vector fields and algebraic closure in ACF_0. Here, "acts like" means that we can develop a dimension that characterizes each model/nice subset as in the above.
- Given an uncountable model, the dimension of its nice subset is exactly the size of the model. Thus, all structures M_1 and M_2 modelling T of the same uncountable size have the same dimension and are isomorphic. †

From this, we can define a nonforking or independence notion for arbitrary subsets of a model. Given $A, B, C \subset M$, we say A is independent from B over C in M, written

$$A \underset{C}{\overset{M}{\perp}} B$$

iff $cl(AC) \cap cl(BC) = cl(C)$. This tells us that A and B contain no overlapping information except what is already contained in C.

It turns out that categorical theories are rare. However, having a nonforking notion is (more) common. This begs the question of what we mean by 'having a nonforking notion.' We want to relate triples as above with some relation that satisfies certain properties: Extension, Uniqueness, Symmetry, etc. Exactly what properties this should satisfy depends on what dividing line the theory satisfies. The prototypical example of a dividing line is stability. On one hand, a theory

[1] This is a bit of a lie. Vector spaces don't quite fit into this because (in terms we will define later) they are in an uncountable language, while ACF_0 is in a countable (even finite) language.

is unstable if there is an infinite definable linear order. Compactness implies that any linear order can be made to appear in a model of this theory, so there are lots of different models; Shelah showed that the order property is enough to give the maximum number of models of any uncountable size. On the other hand, if the theory is stable, then not only is there no infinite definable linear order, but there aren't any more types over a set than elements of the set and their are several equivalent definitions of nonforking that satisfy some very nice properties.

This dual behavior (chaotic on one side, well-behaved on the other, and nothing in between) is emblematic of dividing lines, and classification theory has proved many great results.

1.2. **Classification theory for Abstract Elementary Classes.** First-order classification theory is very successful, but there are many, many classes that aren't first-order axiomatizable. Compactness is a blessing and a curse here: it does a lot of work in model theory, but it means that you can't specify a particular infinite structure in the background, like you might want to do with ω. So this suggests moving to do nonelementary model theory (or just being okay with not dealing with those structures, but that's not the path we take here).

We will discuss the what and the why in detail soon, but the object that we study are Abstract Elementary Classes (AECs for short). These were introduced by Shelah [She87a][2] and cover a much broader range of examples. However, working in AECs is *hard*. We give here a spattering of results.

Fact 1.2. *Let $\mathcal{K}$ be an AEC.*

(1) *If $\mathcal{K}$ has amalgamation and is categorical in λ^+ above*
$$H_2 := \lambda^+ \geq \beth_{\left(2^{\beth_{(2^{LS(\mathcal{K})})^+}}\right)^+}$$
then $\mathcal{K}$ is categorical in every χ such that $H_2 \leq \chi \leq \lambda^+$. [She99]

(2) *If*
- $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$*;*
- *$\mathcal{K}$ is categorical in λ and λ^+; and*
- $0 < I(\mathcal{K}, \lambda^{++}) < \mu_{unif}(\lambda^{++}, 2^{\lambda^+}) \approx 2^{\lambda^+}$,

then $\mathcal{K}$ has a good λ^+-frame. In particular, $\mathcal{K}$ has a model of size λ^{+++}. [She09b, Theorem VI.0.2], slightly weaker version in [She01]

(3) *If $\lambda > \mu \geq LS(\mathcal{K})$ and*
- *$\mathcal{K}$ has no maximal models;*
- *$\mathcal{K}$ is categorical in $\lambda > \beth_{(2^{LS(\mathcal{K})})^+}$; and*
- *GCH*

then $\mathcal{K}$ has unique limit models in μ.

The problem is that the bare bones definition of AECs gives one so little to work with that more power is needed to get anything done.

1.3. **Classification theory for tame Abstract Elementary Classes.** (Hey, that's the name of the course!)

Tameness was introduced by Grossberg and VanDieren in [GV06b].

Definition 1.3. *Let $\mathcal{K}$ be an AEC and $\kappa \geq LS(\mathcal{K})$. $\mathcal{K}$ is* κ-tame *iff for every $M \in \mathcal{K}$ and Galois types $p, q \in gS(M)$, if $p \neq q$, then there is $M_0 \prec M$ of size κ such that $p \upharpoonright M \neq q \upharpoonright M$.*

[2] Most references to this paper will instead be to the revised and more accessible [She] (at least more accessible to owners of [She09b]). However, the results are from the original.

We don't know what Galois types (or AECs) are, but this is asserting a locality property for 'descriptions' of elements. If a and b 'look different' to a model M, then this must be because there's a small submodel of M that already sees that they are different. In first-order, this holds easily because types are sets of formulas, so looking at the formula they differ on gives a finite tuple that already sees there difference. Unfortunately, there's no underlying syntax to Galois types (or the underlying syntax is second order, which is a whole mess), so we don't get this for free.

There's now a wide variety of locality properties for types (locality, type shortness, etc.) and parameterizations, but tameness was the first and most popular. Tameness immediately gives some nicer results (assuming you're willing to say the assumption of tameness and a monster model is nice).

Fact 1.4. *Suppose $\mathcal{K}$ is an AEC with amalgamation and no maximal models that is κ-tame.*

(1) *If $\mathcal{K}$ is categorical in λ^+ above $LS(\mathcal{K})^+ + \kappa$, then $\mathcal{K}$ is categorical in every $\chi \geq \lambda^+$. [GV06c, GV06a]*
(2) *If $\mathcal{K}$ is categorical in λ such that $\operatorname{cf} \lambda > \kappa$ and $\lambda > \kappa = \beth_\kappa$, then $\mathcal{K}$ has a good $\geq \lambda$-frame. [Vas16b, Vas16a][3]*

One reaction to this is that this assumption of tameness is too convenient and that it must be rare. However, the available evidence (which is granted not a ton) points to the opposite, that many natural nonelementary classes are in fact tame and there's some emerging evidence that (in certain contexts) tameness may even act like a dividing line...

2. Introduction

We wish to extend classification theory from first-order model theory to nonelementary classes[4]. However, "nonelementary classes" is not a technical description. It might mean "any class that is not elementary," but this is too broad of a definition to work in: we could arbitrarily restrict to a single model or exclude isomorphism types. The resulting classes would be hard to "do model theory in." Instead, we will use the notion of *Abstract Elementary Classes* (often shortened to AECs) as our framework. Before giving the definition (see Definition 2.5), we examine several natural subclasses. We list several standard logics (and where they come from, if possible). For each logic $\mathcal{L}$ listed (or any logic), we could form a nonelementary class by fixing a language τ and $\mathcal{L}(\tau)$-theory T and considering

$$\mathcal{K} = \operatorname{Mod} T := \{M \mid M \text{ is a } \tau\text{-structure that models } T\}$$

Example 2.1.

(1) $\mathbb{L}_{\lambda,\omega}$ *adds $< \lambda$-sized conjunctions and disjunction. This allows expression of many non-first order concepts, like a group being locally finite or an ordered ring being Achimedean. We insist that all conjunctions still have finitely many free variables, so, e. g.,*

$$\bigwedge_{n<\omega} x_{n+1} < x_n$$

is not an $\mathbb{L}_{\omega_1,\omega}$ formula. Countable fragments (see below) of $\mathbb{L}_{\omega_1,\omega}$ are the most well-studied of this, see Keisler [Kei71].

(2) $\mathbb{L}(Q_\alpha)$ *adds the quantifier $Q_\alpha x \phi(x, \mathbf{y})$, which is interpreted as "there are at least $\aleph_\alpha$-many x such that $\phi(x, \mathbf{y})$ holds." $\mathbb{L}(Q_0)$ is actually $\mathbb{L}_{\omega_1,\omega}$-definable, but $\mathbb{L}(Q_1)$ is not $\mathbb{L}_{\infty,\omega} := \cup_{\alpha \in \boldsymbol{ON}} \mathbb{L}_{\alpha,\omega}$-definable. Q_1 is the most often used*

[3]Sebastien pointed out that "...such that $\operatorname{cf} \lambda > \kappa$ and $\lambda > \kappa = \beth_\kappa \ldots$" can be removed. See [BV17b, Theorem 5.5.2].

[4]In a common misuse, we refer to $\mathbb{L}_{\omega,\omega}$ as first-order logic and extensions such as $\mathbb{L}_{\omega_1,\omega}$ as non-first-order.

(3) $\mathbb{L}(Q^{MM}_\alpha)$ *adds a "Ramsey version" of* Q_α *due to Magidor and Malitz [MM77]. For each* $n < \omega$, $Q^{MM,n}_\alpha x_1, \ldots, x_n \phi(x_1, \ldots, x_n, \mathbf{y})$ *is interpreted as "there is a set* X *of size* $\aleph_\alpha$ *such that for all* $z_1, \ldots, z_n \in X$, $\phi(z_1, \ldots, z_n, \mathbf{y})$ *holds."*

(4) $\mathbb{L}(Q^{er}_\alpha)$ *adds the quantifier* $Q^{er}_\alpha xy\phi(x, y, \mathbf{z})$*, which is interpreted as "*$\phi(x, y, \mathbf{z})$ *defines an equivalence relation on its domain with at least* $\aleph_\alpha$*-many equivalence classes."*

(5) $\mathbb{L}(Q^{cof}_\alpha)$ *adds the quantifier* $Q^{cof}_\alpha xy\phi(x, y, \mathbf{z})$*, which is interpreted as "*$\phi(x, y, \mathbf{z})$ *defines a linear order on its domain with cofinality* α*." This was introduced by Shelah [She75] and is compact!*

(6) $\mathbb{L}(Q^{WF})$ *adds the quantifier* $Q^{WF}xy\phi(x, y, \mathbf{z})$*, which is interpreted as "*$\phi(x, y, \mathbf{z})$ *defines a well-founded relation."*

(7) $\mathbb{L}_{\lambda,\kappa}$ *extends* $\mathbb{L}_{\lambda,\omega}$ *by allowing* $< \kappa$*-ary relations and functions in the language and by permitting the quantification over* $< \kappa$*-many variables at once.*

(8) $\mathbb{L}(I)$ *adds the Härtig 'equicardinality' quantifier [Här62]* $Ix\phi(x, \mathbf{y})\psi(x, \mathbf{y})$*, which is interpreted as "there are the same number of* x *so* $\phi(x, \mathbf{y})$ *holds as* x *so* $\psi(x, \mathbf{y})$ *holds."*

(9) $\mathbb{L}(aa)$ *adds the second order quantifier aa, 'almost all.' In addition to allowing second order variables and the atomic formula* $x \in s$*, the quantifier aa* $s\phi(s, \mathbf{x}, \mathbf{s})$ *is interpreted to mean there is a club of countable subsets of the universe that satisfy the formula.*

(10) $\mathbb{L}^2$ *is second-order logic, where we allow quantification over subsets of cartesian powers of the universe.*

(11) $\mathbb{L}^{s,\Sigma_n}$ *is sort logic, introduced by Väänänen [Vää79]. This allows one to quantify over arbitrary sets. This is very powerful, and to make it definable,* $\mathbb{L}^{s,\Sigma_n}$ *restricts to* n*-many sort quantifiers.*

Exercise 2.2. *Show that* $\mathbb{L}(Q^{WF})$ *is a proper extension of* $\mathbb{L}_{\omega,\omega}$ *and is expressible in both* $\mathbb{L}_{\omega_1,\omega_1}$ *and* $\mathbb{L}^2$*.*

Thus, logics beyond $\mathbb{L}_{\omega,\omega}$ allow us to give a first-pass at what we mean by a nonelementary class that we can do model theory in. However, classification theory is about more than just the class of models under consideration. The notion of *elementary substructure* is a powerful and useful concept in model theory. Each logic above comes with a notion of elementarity and, in many cases, several notions.

Example 2.3. *Fix a language* τ*.*

(1) $\mathcal{F} \subset \mathcal{L}(\tau)$ *is called a* fragment *iff it is closed under subformulas and interchanging free variables.*

(2) *A fragment* $\mathcal{F} \subset \mathcal{L}(\tau)$ *is* elementary *iff it is closed under the first-order operations.*[5]

(3) *We define the notion of* $\mathcal{F}$*-elementary substructure for fragments* $\mathcal{F}$ *of various logics:*

  (a) *If* $\mathcal{F} \subset \mathbb{L}_{\lambda,\omega}(\tau)$*, then* $M \prec_{\mathcal{F}} N$ *iff for every* $\phi(\mathbf{x}) \in \mathcal{F}$ *and* $\mathbf{m} \in M$,
  $$M \vDash \phi(\mathbf{m}) \iff N \vDash \phi(\mathbf{m})$$

  (b) *If* $\mathcal{F} \subset \mathbb{L}_{\lambda,\omega}(Q_\alpha)(\tau)$*, then* $M \prec_{\mathcal{F}} N$ *iff for every* $\phi(\mathbf{x}) \in \mathcal{F}$ *and* $\mathbf{m} \in M$,
  $$M \vDash \phi(\mathbf{m}) \iff N \vDash \phi(\mathbf{m})$$
  and, if $\neg Q_\alpha x\phi(x, \mathbf{y}) \in \mathcal{F}$ *and* $\mathbf{m} \in M$ *such that* $M \vDash \neg Q_\alpha x\phi(x, \mathbf{m})$*, then* $\phi(M, \mathbf{m}) = \phi(N, \mathbf{m})$.

  (c) *If* $\mathcal{F} \subset \mathbb{L}_{\lambda,\omega}(Q^{cof}_\alpha)(\tau)$*, then* $M \prec_{\mathcal{F}} N$ *iff for every* $\phi(\mathbf{x}) \in \mathcal{F}$ *and* $\mathbf{m} \in M$,
  $$M \vDash \phi(\mathbf{m}) \iff N \vDash \phi(\mathbf{m})$$

[5] Note that some sources, especially Keisler [Kei71], use fragment for what we call elementary fragment. However, as we will see the weaker notion of a fragment not closed under first-order operations is more versatile and useful.

and, if $Q^{cof}_\alpha xy\phi(x, y, \mathbf{z}) \in \mathcal{F}$ and $\mathbf{m} \in M$ such that $M \vDash Q^{cof}_\alpha xy\phi(x, y, \mathbf{m})$, then any cofinal α sequence in M is also cofinal in N.

(d) *If $\mathcal{F} \subset \mathbb{L}^2(\tau)$, then $M \prec_\mathcal{F} N$ iff there is an extension $s \subset M \mapsto s^N \subset N$ such that $s \cap N = M$ and, for every $\phi(\mathbf{x}, \mathbf{X}) \in \mathcal{F}$ and $\mathbf{m} \in M, \mathbf{s} \subset M$, we have*

$$M \vDash \phi(\mathbf{m}, \mathbf{s}) \iff N \vDash \phi(\mathbf{m}, \mathbf{s}^N)$$

Exercise 2.4. *Show that for any $T \subset \mathbb{L}_{\lambda,\omega}(Q^1, \ldots, Q^n)$ (where Q^i is an arbitrary quantifier), there is a minimal (elementary) fragment containing T.*

Thus, to pin down a nonelementary class that we can do model theory in, we want to specify a) a class of structures and b) a special substructure relation between them that c) satisfies certain closure axioms. We will use the axioms of Abstract Elementary Classes, first given by Shelah [She87a,] in 1987, although Grossberg reports that paper had been circulating for many years before.

Definition 2.5. *We say that $(\mathcal{K}, \prec_\mathcal{K})$ is an Abstract Elementary Class iff*

(1) *There is some language $\tau = \tau(K)$ so that every element of $\mathcal{K}$ is an τ-structure;*
(2) *$\prec_\mathcal{K}$ is a partial order on $\mathcal{K}$ that respects τ-isomorphism and refines $\subset_\tau$;*
(3) (Coherence) *if $M_0, M_1, M_2 \in \mathcal{K}$ with $M_0 \prec_\mathcal{K} M_2$, $M_1 \prec_\mathcal{K} M_2$, and $M_0 \subset_\tau M_1$, then $M_0 \prec_\mathcal{K} M_1$;*
(4) (Tarski-Vaught axioms) *suppose $\langle M_i \in \mathcal{K} : i < \alpha\rangle$ is a $\prec_\mathcal{K}$-increasing continuous chain, then*
    (a) *$\cup_{i<\alpha} M_i \in \mathcal{K}$ and, for all $i < \alpha$, we have $M_i \prec_\mathcal{K} \cup_{i<\alpha} M_i$; and*
    (b) *if there is some $N \in \mathcal{K}$ so that, for all $i < \alpha$, we have $M_i \prec_\mathcal{K} N$, then we also have $\cup_{i<\alpha} M_i \prec_\mathcal{K} N$; and*
(5) (Löwenheim-Skolem-Tarski[6] number) *$LS(\mathcal{K})$ is the minimal infinite cardinal $\lambda \geq |\tau(\mathcal{K})|$ such that for any $M \in \mathcal{K}$ and $A \subset |M|$, there is some $N \prec_\mathcal{K} M$ such that $A \subset |N|$ and $\|N\| \leq |A| + \lambda$.*

Unfortunately, not all of the above logics give rise to AECs. However, many do.

Exercise 2.6. *If $\mathcal{F}$ is a fragment of $\mathbb{L}_{\lambda,\omega}(Q_\alpha)$ and $\psi \in \mathcal{F}$, then $(Mod\ \psi, \prec_\mathcal{F})$ is an AEC. What is it's Löwenheim-Skolem-Tarski number?*

Here are a few concrete examples based on the previous exercise. We're still using a lot terms we haven't defined.

Example 2.7.

(1) **Torsion groups:** *Set*

$$T_{tor} = T_{grp} \cup \{\forall x \bigvee_{n<\omega} x^n = e\}$$

*Then T is the $\mathbb{L}_{\omega_1,\omega}$-theory of torsion groups. $(Mod\ T, \subset)$ is an AEC with $LS(\mathcal{K}) = \aleph_0$. It is a universal class, which implies it is $< \omega$-tame (see Section **??**).*

However, we probably can't do much classification theory; the class is too broad and subgroup is too weak. Fix a PID R and a torsion module M. Set

$$T_M = Th_{\mathbb{L}_{\omega,\omega}}(M) \cup \{\forall x \bigvee_{r \in R_{\neq 0}} r \cdot x = 0\}$$

[6]This was originally and still often is called the Löwenheim-Skolem number, which explains the use of 'LS($\mathcal{K}$).' However, a more refined reading of the development of this theorem in the first-order context suggests including Tarski, as seen in Magidor and Väänänen [MV11].

This is a $\mathbb{L}_{|R|^+ + \aleph_0, \omega}$-theory, so $\mathcal{K}_{tor} = (Mod\ T_M, \prec)$ is an AEC with $LS(\mathcal{K}_{tor}) = |R| + \aleph_0$. [Bona, Section 5] discusses these classes. They have amalgamation, joint embedding, have no maximal models or have one model, are $< \omega$-tame, and are stable (mainly because the first-order theory is stable).

(2) **Locally finite groups:** *Set*

$$T_{lfg} = T_{grp} \cup \{\forall x_1, \dots, x_n \bigvee_{finite G_0} \text{``}\langle x_1, \dots, x_n\rangle \cong G_0'' \mid n < \omega\}$$

where "$\langle x_1, \dots, x_n\rangle \cong G_0''$ is an abbreviation for the first-order sentence saying the group generated by $x_1, \dots, x_n$ is isomorphic to the finite group G_0. Set $\mathcal{K}_{lfg} = (Mod\ T_{lfg}, \subset)$. This is a universal class as well. The existentially closed models of this class are the universal locally finite groups, see Macintyre and Shelah [MS76] and Shelah [She17] for more on these. These classes are $\aleph_0$-categorical, fail amalgamation, and are $< \omega$-tame.

(3) **Zilber's Pseudo-exponential fields:** *Expand an algebraically closed field K with a surjective homomorphism $exp : K^+ \to K^\times$ and additionally require various properties, especially*
- *the kernel of exp is an infinite cyclic group; this allows $\mathbb{Z}$ to be definable;*
- *a Schanuel's conjecture-like property holds; and*
- *the set of roots of exponential polynomials over a finite set is countable.*

This requires $\mathbb{L}_{\omega_1, \omega}(Q_1)$ for an axiomatization. This class is totally categorical, has amalgamation, is $< \omega$-tame. See [Bonf] for specifics.

More generally, these are an example of quasiminimal classes introduced by Zilber []. These classes have a quasiminimal closure (like exponential closure in the case above) that allows one to develop a theory of dimension as with vector spaces.

One can also use $\mathbb{L}_{\lambda^+, \omega}$ to pin down structures of size λ. This means that you can form AECs by taking an elementary class and fixing a predicate. This means, e. g., the class of algebraically closed valued fields with a specified value group is an AEC as are metric spaces (although there are easier ways to accomplish this [BBHU08].

Example 2.8. *We can also make the cofinality quantifiers into an AEC with a little more work. We have to restrict to what I call positive, deliberate uses. Fix a language τ and a theory $T \subset \mathbb{L}(Q^{cof}_\alpha)$ that doesn't allow negations to be used after a cofinality quantifier (this explains the positive part). Close τ to τ^+ so that whenever $\phi(x, y, \mathbf{z})$ is in $\mathbb{L}(Q^{cof}_\alpha)(\tau^+)$, so is $R_\phi(\mathbf{z})$. Form the $\mathbb{L}(\tau)$-theory T^+ that inductively replaces each instances of $Q^{cof}_\alpha xy\phi(x, y, \mathbf{z})$ with $R_\phi(\mathbf{z})$. Now consider τ^+ structures that are models of*

$$T^* := T^+ \cup \left\{\forall \mathbf{z} \left(R_\phi(\mathbf{z}) \to Q^{cof}_\alpha xy\phi(x, y, \mathbf{z})\right) \mid \phi(x, y, \mathbf{z}) \in \mathbb{L}(Q^{cof}_\alpha)(\tau^+)\right\}$$

Then $\left(Mod\ T^*, \prec_{\mathbb{L}(Q^{cof}_\alpha)}\right)$ is an AEC with Löwenheim-Skolem number $|\alpha| + \aleph_0$. Moreover, every model of T can be expanded to a model of T^*. In fact, they can be expanded in a number of ways. For each, $\phi(x, y, \mathbf{m})$ that defines a linear order of cofinality α, a choice must be made as to wether or not to enforce the cofinality of that linear order. Moreover, this choice must be the same in any $\prec_{\mathbb{L}(Q^{cof}_\alpha)}$-submodel that contains $\mathbf{m}$.

The positivity is necessary because we demand closure under all increasing chains. If we allowed a negative use to say some linear order doesn't have cofinality ω, then there couldn't be an increasing chain of models that add an extra point at the end: no matter how many points they added, in the union the order would have cofinality ω. The move to R_ϕ is necessary because we similarly worry that a linear order might "accidentally" have cofinality ω after a countable *union. If full elementarity was required, this would force that to hold on every submodel. Instead, we choose to be deliberate about which definable linear orders that we specify the cofinality of.*

A common theme amongst the logics that define AECs is that they are all expressible in a fragment of $\mathbb{L}_{\kappa,\kappa}$ that has finitary Skolem functions. Moreover, the (perhaps) arbitrary (or self-serving) choice of which notion of elementarity should be used can be justified by looking at the fragment necessary to express those quantifiers and using elementarity according to that fragment.

For instance, consider $\mathbb{L}(Q_1)$, logic with the quantifier "there exists uncountably many." Q_1 are expressible in $\mathbb{L}_{\omega_1,\omega_1}$. For reasons that will become apparent later (see Section **??**), we give positive existential characterizations of both positive and negative instances of Q_1 in $\mathbb{L}_{\omega_2,\omega_2}$:

$$Q_1 x\phi(x,\mathbf{y}) \quad \text{is equivalent to} \quad \exists\{x_i \mid i<\omega_1\}\left(\bigwedge_{i<\omega_1}\phi(x_i,\mathbf{y})\wedge\bigwedge_{i\neq j<\omega_1}x_i\neq x_j\right)$$

$$\neg Q_1 x\phi(x,\mathbf{y}) \quad \text{is equivalent to} \quad \exists\{x_i \mid i<\omega\}\forall z\left(\phi(z,\mathbf{y})\to\bigvee_{i<\omega}z=x_i\right)$$

Skolem function for these schema remain finitary as they only depend on $\mathbf{y}$. Further, elementarity for the first schema follows from elementarity for ϕ (ω_1-sized sets remain ω_1-sized) and elementarity for the second schema requires (elementarity for ϕ and) that any witness to the countability in the small model remains a witness in the larger model. This is exactly the condition that $M \vDash$ "$\neg Q_1 x\phi(x,\mathbf{a})$" implies $\phi(M,\mathbf{a}) = \phi(N,\mathbf{a})$.

Exercise 2.9. *Find strong substructure relations for $\mathbb{L}(Q_\alpha^{MM})$ and $\mathbb{L}(Q_\alpha^{er})$. Hint: Find a way to express them in $\mathbb{L}_{\infty,\infty}$.*

We have started with a motivation coming from various extensions of first-order logic, but moved a purely semantic axiomatization of "nonelementary class we can do model theory in." An obvious question becomes wether there is some master logic $\mathbb{L}_{AEC}$ such that $(\mathcal{K},\prec_K)$ is an AEC iff it is (Mod $T,\prec_{\mathcal{F}}$) for a $\mathbb{L}_{AEC}$-theory T and a fragment $\mathcal{F}$ containing it. This question is currently open and we return to a discussion of it in Subsection **??**. Before then, keep an eye open for Shelah's Presentation Theorem 3.25, which sheds some light on the question.

3. Building a toolbox

3.1. **Putting an eye on the prize.** Now that we have an idea of what AEC's are, what do we want to do with them? The answer is 'some classification theory.' This result is very vague, so we have a test question. Morley's Theorem [Mor65a] is often seen as the "birth of modern model theory" (where modern model theory means classification theory), so we use this to base our test question on.

Theorem 3.1 (Morley [Mor65a], Baldwin-Lachlan [BL71], Shelah [She74])**.** *Let T be a first-order theory. If T is κ-categorical for* some $\kappa > |T|$, *then T is κ-categorical for* every $\kappa > |T|$.

Morley proved this for $|T| = \aleph_0$, Baldwin-Lachlan refined this proof to give more information, and Shelah proved this for uncountable theories.

The initial hope might be that this could be generalized to AECs by replacing $|T|$ with $\mathrm{LS}(\mathcal{K})$, but this turns out to fail. Consider the following example due (I think) to Kueker.

Example 3.2. *Fix a cardinal λ. Set $\mathcal{K}^\lambda$ to be the AEC consisting $(X,<)$, where $(X,<)$ is well-founded and it's order-type is in $[\lambda,\lambda^+]$. Set $(X,<)\prec^\lambda(Y,<)$ iff $(X,<)$ is an initial segment of $(Y,<)$. Then*

(1) *$\mathcal{K}^\lambda$ is λ^+-categorical, as all λ^+-sized elements are isomorphic to $(\lambda^+, \in)$; and*
(2) *$\mathcal{K}^\lambda$ is not μ-categorical for any $\mu \geq \lambda^{++}$, since it has no models of those size.*

The situation is actually much worse. Here is a fact that we will discuss later:

Fact 3.3. *Fix $\alpha < \lambda^+$. There is an AEC $\mathcal{K}^\alpha$ with $LS(\mathcal{K}^\alpha) = \lambda$ that has models of size $\beth_\alpha$, but no larger.*

The maximum α can actually be much larger than λ^+ and corresponds to the ordinals that can be 'pinned down' by an AEC with $\mathrm{LS}(\mathcal{K}) = \lambda$. The best bound in general is $\left(2^{\mathrm{LS}(\mathcal{K})}\right)^+$.

Exercise 3.4. *Show that for every $\alpha < \lambda^+$, there is an AEC $\mathcal{K}^\alpha$ that is definable by one of our logics such that every model in it has order-type α.*

This fact means that the threshold cardinal has to be at least $\beth_\delta$, where δ is the first ordinal not pinned down by AECs with $\mathrm{LS}(\mathcal{K}) \leq \lambda$. This leads us to several revised categoricity conjectures:

Conjecture 3.5 (Shelah's Categoricity Conjecture(s))**.** (1) *Suppose $\psi \in \mathbb{L}_{\omega_1,\omega}(\tau)$. If Mod ψ is κ-categorical for* some *$\kappa \geq \beth_{\omega_1}$, then Mod ψ is κ-categorical for* every *$\kappa \geq \beth_{\omega_1}$.*
(2) *Suppose $\mathcal{K}$ is an AEC with $LS(\mathcal{K}) = \lambda$. If $\mathcal{K}$ is κ-categorical for* some *$\kappa \geq \beth_{\left(2^{LS(\mathcal{K})}\right)^+}$, then $\mathcal{K}$ is κ-categorical for* every *$\kappa \geq \beth_{\left(2^{LS(\mathcal{K})}\right)^+}$.*
(3) *For each λ, there is a cardinal μ_λ such that: suppose $\mathcal{K}$ is an AEC with $LS(\mathcal{K}) = \lambda$. If $\mathcal{K}$ is κ-categorical for* some *$\kappa \geq \mu_\lambda$, then $\mathcal{K}$ is κ-categorical for* every *$\kappa \geq \mu_\lambda$.*
(4) *Each of the above strengthened to assume that the initial categoricity cardinal is a successor.*

These are variously called 'Shelah's Categoricity Conjecture.' (2) is probably the one I would most often call *the* Shelah's Categoricity Conjecture. (1) is denoted by '...for $\mathbb{L}_{\omega_1,\omega}$,' (3) is denoted by 'Shelah's Eventual...,' and adding the modifier in (4) is denoted by '...for successors.'

The inclusion of 'Shelah's Eventual Categoricity Conjecture' is in part meant to show how far we have to go. Even if you're allowed to pick the threshold cardinal, then this isn't known. Here are three approximations:

Theorem 3.6.
(1) *If $\mathcal{K}$ has amalgamation and is categorical in λ^+ above*

$$H_2 := \lambda^+ \geq \beth_{\left(2^{\beth_{\left(2^{LS(\mathcal{K})}\right)^+}}\right)^+}$$

then $\mathcal{K}$ is categorical in every χ such that $H_2 \leq \chi \leq \lambda^+$. [She99] (See Theorem 5.68.)
(2) *If $\mathcal{K}$ has amalgamation, no maximal models, is κ-tame, and is categorical in λ^+ above $LS(\mathcal{K})^+ + \kappa$, then $\mathcal{K}$ is categorical in every $\chi \geq \lambda^+$. [GV06c, GV06a] (See Theorem 5.67.)*
(3) *If there are class-many almost strongly compact cardinals, then Shelah's Eventual Categoricity Conjecture for Successors is true. [Bon14] (See Theorem 5.69.)*

I've underlined some undefined terms in the above. The weird cardinal in (1) is call the second Hanf number of $\mathrm{LS}(\mathcal{K})$, with the Hanf number of λ being $\beth_{(2^\lambda)^+}$. Tameness is, of course, the key property that we wish to study in this course.

Our goal is going to be to prove these three theorems and use them as motivation to develop some of the basics around AECs (EM models, Galois types, etc.). Our proof of the first two will follow the presentation in Baldwin [Bal09]. These theorems were proved in the order presented and each use ideas from the ones that came before, e. g., most (but not all!) of the heavy lifting for my theorem was done by Shelah and Grossberg-VanDieren.

3.2. **A monstrous model.** In first-order model theory, we often work inside a 'monster model' $\mathfrak{C}$ (less provocatively called a universal domain) that is highly saturated. In ideal cases, this is built from an infinitary Fraïssé process which uses that first-order theories have several nice properties: amalgamation, joint embedding, and no maximal models.

However, AECs can fail to have these properties.

Definition 3.7. *For a set of cardinals $\mathcal{F}$, $\mathcal{K}_\mathcal{F} := \{M \in \mathcal{K} \mid \|M\| \in \mathcal{F}\}$. If $\mathcal{F} = \{\lambda\}$ is a singleton, we just write λ.*
$\mathcal{K}$ has the λ-amalgamation property iff for all $M_0, M_1, M_2 \in \mathcal{K}_\lambda$ with $M_0 \prec M_1, M_2$, there is $N \succ M_2$ with $f : M_1 \to_{M_0} N$.
$\mathcal{K}$ has the amalgamation property iff for all $M_0, M_1, M_2 \in \mathcal{K}_\lambda$ with $M_0 \prec M_1, M_2$, there is $N \succ M_2$ with $f : M_1 \to_{M_0} N$.

Example 3.8.

(1) *Elementary classes have amalgamation.*
(2) *The AEC of locally finite groups with subgroup has $\aleph_0$-amalgamation but does not have amalgamation. [MS76]*
(3) *Let $\tau = \{E, P_n \mid n < \omega\}$ and $\psi \in \mathbb{L}_{\omega_1,\omega}(\tau)$ be*

$$\text{“}E\text{ is an equivalence relation”} \wedge \forall x \bigwedge_{n<\omega} P_{n+1}(x) \to P_n(x) \wedge$$
$$\bigwedge_{n<\omega} \exists x_1, x_2(P_n(x_1) \wedge P_n(x_2) \wedge \neg P_{n+1}(x_1) \wedge$$
$$\neg P_{n+1}(x_2) \wedge \neg x_1 E x_2 \wedge \forall y(P_n(y) \wedge \neg P_{n+1}(y) \to y = x_1 \vee y = x_2)) \wedge$$
$$\forall x, y \left(\left(\bigwedge_{n<\omega} P_n(x) \wedge P_n(y) \right) \to xEy \right)$$

Then Mod ψ fails $\aleph_0$-amalgamation, but has μ-amalgamation for every $\mu > \aleph_0$.
(4) *Valued fields with a fixed value group and subfield have amalgamation. [Bond]*

Proposition 3.9. *$\mathcal{K}$ has the amalgmation property iff it has the λ-amalgamation property for all $\lambda \geq LS(\mathcal{K})$.*

The proof of this proposition uses the following very nice technique.

Lemma 3.10. *For every $M \in \mathcal{K}_{>LS(\mathcal{K})}$, there is an $\prec$-increasing, continuous $\langle M_i \mid i < cf\, \|M\| \rangle$ such that $M_i \prec M$, $\|M_i\| < \|M\|$, and $\cup_{i<cf\,\|M\|} M_i = M$. Moreover, if $\langle \kappa_i \mid i < cf\, \|M\| \rangle$ is increasing, continuous and cofinal in $\|M\|$, then you can choose this sequence so $\|M_i\| = \kappa_i + LS(\mathcal{K})$.*

Proof: Find $A_i \subset M$ such that $|A_i| = \kappa_i$, $\cup_{i<\mathrm{cf}\,\|M\|} A_i = M$. We build M_i by induction to contain $\cup_{j<i} A_j$.

- $i = 0$: Use the Löwenheim-Skolem-Tarski axiom to find $M_0 \prec M$ of size LS($\mathcal{K}$) such that $A_0 \subset M_0$.
- i limit: Take unions; we have to do this to get continuity. By the Tarski-Vaught Chain axiom, this union is a $\prec$-upper bound of the sequence and is a strong substructure of M.
- $i = j + 1$: Use the L$\acute{}$ owenheim-Skolem-Tarski axiom to find $M_i \prec M$ of size LS($\mathcal{K}$) $+ \kappa_i$ that contains $A_j \cup M_j$. By coherence, $M_j \prec M_i$.

Then the sequence is as desired.

Sequences as in the above lemma are called *resolutions*. Generally, this will mean a increasing, continuous sequence that unions to the model in question so each member is of smaller size. See Exercise 3.27 for a generalization.

Proof of Proposition 3.9: We show by induction that $\mathcal{K}_{[\mathrm{LS}(\mathcal{K}),\chi]}$ has amalgamation. If $\chi = \mathrm{LS}(\mathcal{K})$, this is by assumption.

Suppose $\chi > \mathrm{LS}(\mathcal{K})$ and $M_0 \prec M_1, M_2$ are an triple from $\mathcal{K}_{[\mathrm{LS}(\mathcal{K}),\chi]}$. If $\|M_0\| = \chi$, then we use the hypothesis. First assume that $\|M_0\|, \|M_1\| < \chi$ and $\|M_2\| = \chi$. Find a resolution $\{M_2^i \mid i < \chi\}$ such that $M_2^0 = M_0$. We build, by induction, increasing continuous $\{N_i \in \mathcal{K}_{<\chi} \mid i < \chi\}$ and $f_i : M_2^i \to N_i$ such that $N_0 = M_1$. We can do this by assumption since all models are smaller than χ. Then $\cup_{i<\chi} f_i : M_2 \to \cup_{i<\chi} N_i$ is the desired amalgam.

Finally, if $\|M_1\| = \|M_2\| = \chi$, then the same technique can be used: resolve M_1 and use the previous paragraph to amalgamate these models. †

Once we have amalgamation, we can define the notion of *Galois types*, which replaces the syntactic notion of type in first-order. These were isolated by Shelah [She87b] in studying universal classes (see Section **??**) and name Galois types by Grossberg [Gro02]. One should note that Shelah prefers the term 'orbital type.'

Definition 3.11. *Let $\mathcal{K}$ be an AEC, $\lambda \geq LS(\mathcal{K})$, and I a set.*

(1) *The set of pretypes of length I is $\mathcal{K}^{3,I}_\lambda := \left\{(\langle a_i \mid i \in I\rangle, A, M, N) \mid M \in \mathcal{K}_\lambda; A \subset M; N \in K_{\lambda+|I|}; M \prec N; a_i \in N\right\}$.*

(2) *Given pretypes over the same domain $\left(\langle a_i^0 \mid i \in I\rangle, A, M, N^0\right)$ and $\left(\langle a_i^1 \mid i \in I\rangle, A, M, N^1\right)$, we say that they are atomically equivalent*

$$\left(\langle a_i^0 \mid i \in I\rangle, M, N^0\right) \sim_{AT} \left(\langle a_i^1 \mid i \in I\rangle, M, N^1\right)$$

iff there is an amalgam N^* with $f_\ell : N^\ell \to N^*$ such that $f_0(a^0) = f_1(a^1)$ and $f_0 \upharpoonright A = f_1 \upharpoonright A$.

(3) *Equivalence of pretypes $\sim$ is the transitive closure of $\sim_{AT}$.*

(4) *Galois types are equivalence classes of pretypes:*

$$gtp\,(\langle a_i \mid i \in I\rangle / A, M; N) = [\langle a_i \mid i \in I\rangle, A, M, N]_\sim$$

(5) *Set $gS^I(A; M) = \{gtp\,(\langle a_i \mid i \in I\rangle / A, M; N) \mid (\langle a_i \mid i \in I\rangle / A, M; N) \in K^{3,I}_{\|M\|}$ and $gS(A; M) = gS^1(A; M)$. If A is a model, we write $gS(M) = \cup_{M \prec N}\, gS(M; N)$.*

(6) *Let $p = gtp\,(\langle a_i \mid i \in I\rangle / A, M; N)$, $I_0 \subset I$, and $M_0 \prec M$ and $A_0 \subset A$ such that $A_0 \subset M_0$. Then*

$$\begin{aligned} p^{I_0} &:= gtp\,(\langle a_i \mid i \in I_0\rangle / A, M; N) \in gS^{I_0}(A; M) \\ p \upharpoonright (A_0, M_0) &:= gtp\,(\langle a_i \mid i \in I\rangle / A_0, M_0; N) \in gS^{I}(A_0; M_0) \end{aligned}$$

Proposition 3.12. *For any $M \in \mathcal{K}_\lambda$, $|gS(M)| \leq 2^\lambda$.*

Proof: There are at most that many isomorphism types in K^3_λ. †

Now that we've defined types, we can define tameness! This is one version, but there are many variation on it which we will discuss in Section 4

Definition 3.13 ([GV06b,])**.** $\mathcal{K}$ *is* κ-tame of length β *iff for all $p, q \in gS^\beta(M)$, if $p \neq q$, then there is $A \subset M$ of size $\leq \kappa$ such that $p \upharpoonright A \neq q \upharpoonright A$.*

If $\beta = 1$, we omit it. We say $\mathcal{K}$ is tame iff it is κ-tame for some κ.

Note that this is standard, but [GV06b] use tame to mean κ-tame for some $\kappa < \beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$. [GV06b, Conjecture 3.4] conjecture they are the same.

In first-order, proofs of categoricity transfer typically work by showing that all models are saturated. Work in AECs is similar, so we need a notion of saturation.

Definition 3.14. (1) *M is* λ-Galois saturated *iff for all $M_0 \prec M$ of size $< \lambda$ and $p \in gS(M_0)$, p is realized in M. M is* Galois saturated *iff it is $\|M\|$-Galois saturated.*
(2) *M is* λ-model homogeneous *iff for all $M_0 \prec M$ of size $< \lambda$ and $M_1 \succ M_0$ with $\|M_0\| = \|M_1\|$, there is $f : M_1 \to_{M_0} M$.*

Exercise 3.15. (1) *M is $\|M\|^+$-Galois saturated iff it has no proper extensions.*
(2) *Let M be λ-model homogeneous, $M_0 \prec M$ of size $< \lambda$ and $M_1 \succ M_0$ of size λ. Show there is $f : M_1 \to_{M_0} M$.*

Under amalgamation, we can build λ-saturated extensions.

Proposition 3.16. *Suppose that $\mathcal{K}$ has amalgamation and $\lambda > LS(\mathcal{K})$.*

(1) *Given $M \in \mathcal{K}$, there is $N \succ M$ that is λ-saturated.*
(2) *Suppose that, for every $M \in \mathcal{K}_\lambda$, $|gS(M)| = \lambda$. Then every $M \in \mathcal{K}_\lambda$ has a saturated extension in $\mathcal{K}_{\lambda^+}$.*

The extra hypothesis in (2) is called λ-*Galois stability* (see Definition 3.36).

Proof: First, we describe a general construction $M \mapsto M^*$ such that M^* realizes every type over a $< \lambda$-sized submodel of M. Let $\{(M_i, p_i) \mid i < \mu\}$ enumerate the pairs so $M_i \prec M$, $\|M_i\| < \lambda$, and $p_i \in \mathrm{gS}(M_i)$. Build an increasing continuous sequence $\{M_i \mid i < \mu\}$ by setting $M_0 = M$ and $M_{i+1} \succ M_i$ realize p_i; this can be done by amalgamation. Then $M^* := \cup_{i<\mu} M_i$ is as desired.

For (1), fix $N \in \mathcal{K}$ and build increasing continuous $\{N_i \mid i < \lambda^+\}$ such that $N_0 = N$ and $N_{i+1} = (N_i)^*$. Set $N^+ := \cup_{i<\lambda^+} N_i \succ N$. Given $M_0 \prec N^+$ of size $< \lambda$, since λ^+ is regular, there is $i_0 < \lambda^+$ such that $M_0 \prec N_{i_0}$. Then every type over M_0 is realized in $N_{i_0+1} \prec N^+$.

For (2), we do essentially the same construction. Note that we don't need to enumerate the different submodels in the general construction and our assumption gives $\mu = \lambda$. Thus, each of the models N_i is of size λ. So the resulting N^+ is λ^+-saturated of size λ^+. †

The following lemma says that Galois saturation and model homogeneity are the same under amalgamation. [Bal09, Theorem 8.14] gives a simpler proof under the assumption of a monster model.

Theorem 3.17 ([She01, Lemma 0.26])**.** *Suppose $\mathcal{K}$ has $< \lambda$-amalgamation. The following are equivalent:*

(1) *M is λ-Galois saturated.*
(2) *M is λ-model homogeneous*

Proof: It is easy to show model homogeneity implies Galois saturation.

Suppose M^+ is λ-Galois saturated, $M \prec M^+$, and $M \prec N$, with $\|M\| = \|N\| = \mu < \lambda$. Enumerate N as $\{a_i \mid i < \mu\}$. We build increasing and continuous $\{N^i_\ell, f_i \mid i \leq \mu, \ell = 0, 1\}$ by induction on $i \leq \mu$ such that

(1) $N^0_1 = M$, $N^0_2 = N$, and $f_0 = \mathrm{id}_N$;
(2) $N^i_1 \prec N^i_2$ in $\mathcal{K}_\mu$;
(3) $f_i : N^i_1 \to M^+$; and
(4) $a_i \in N^{i+1}_1$.

In the end, we have the following diagram:

$$\begin{tikzcd}
N_2^\mu & & & M^+ \\
N \arrow[u] \arrow[r] & N_1^\mu \arrow[ul] \arrow[urr, "f_\mu"] & & \\
& M \arrow[ul] \arrow[u] \arrow[uurr] & &
\end{tikzcd}$$

The base case and limit steps are set by the above.

Suppose $i = j + 1$. If $a_j \in N_1^j$, then we are done. Otherwise, set $M_1^i = f_i(N_1^i)$. We can amalgamate N_2^j and M_1^j over N_1^j into $M^* \succ M_1^j$ with $g_j : N_2^j \to M^*$ extending f_j (see the diagram at the end of the proof). By the λ-Galois saturation of M^+, gtp $\left(g_j(a_j)/M_1^j; M^*\right)$ is realized in M^+. Find $M^- \prec M^+$ of size μ, $b \in M^-$ and $N^* \succ M^-$ and $h : M^* \to_{M_1^j} N^*$ such that $h\left(g_j(a_j)\right) = b$. Now do some renaming to find $N_2^{j+1} \succ N_2^j$ and $f : N_2^{j+1} \cong N^*$ extending $h \circ g_j$. Set $N_1^{j+1} = f^{-1}(M^-)$; this is as desired.

$$\begin{tikzcd}
N_2^{j+1} \arrow[drr, "f", "\cong"'] & & & \\
N_2^j \arrow[u] \arrow[r, "g_j"] & M^* \arrow[r, "h"] & N^* & \\
N_i^j \arrow[u] \arrow[r, "f_j"] & M_1^j \arrow[u] \arrow[r] & M^- \arrow[u] \arrow[r] & M^+
\end{tikzcd}$$

After the construction, $N \subset N_1^\mu$. By coherence, $N \prec N_1^\mu$, so $f_\mu \upharpoonright N : N \to_M M^+$. †

Exercise 3.18 ([Bon17, Proposition 5.2]). *We defined Galois saturation in terms of 1-types. Show that λ-Galois saturated models are also λ-Galois saturated for $< \lambda$-length types.*

There are some additional structural properties we will consider.

Definition 3.19. *Let $\mathcal{K}$ be an AEC.*

(1) *$\mathcal{K}$ has the* λ-joint emebedding property *iff for every $M_0, M_1 \in K_\lambda$, there is $N \in \mathcal{K}_\lambda$ and $f_\ell : M_\ell \to N$ for $\ell = 0, 1$.*
(2) *$\mathcal{K}$ has* no maximal models *iff for every $M \in \mathcal{K}$ there is $N \in \mathcal{K}$ so $M \precneqq N$.*
(3) *$\mathcal{K}$ has* arbitrarily large models *iff for every λ, there is $M \in \mathcal{K}_{>\lambda}$.*

All of these will be necessary to build a monster model. Why do we focus on just having amalgamation so much?

Exercise 3.20. *Suppose $\mathcal{K}$ is categorical in $\lambda \geq LS(\mathcal{K})$.*

(1) *If $\mathcal{K}$ has amalgamation, then $\mathcal{K}_{\geq\lambda}$ has joint embedding.*
(2) *If $\mathcal{K}$ has no maximal models, then $\mathcal{K}_{\leq\lambda}$ has joint embedding.*

In the presence of joint embedding, having no maximal models and arbitrarily large models are equivalent.

So amalgamation and categoricity in a sufficiently big cardinal is enough to get a monster model; sufficiently large means $\geq \beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$; see Theorem 3.23.

Theorem 3.21. *Let $\mathcal{K}$ be an AEC and $\lambda > LS(\mathcal{K})$ with amalgamation, joint embedding, and no maximal models. Then $\mathcal{K}$ has a λ-monster model $\mathfrak{C}$; that is, a model that is*

(1) *λ-universal;*
(2) *λ-model homogeneous; and*
(3) *if $M \prec \mathfrak{C}$ of size $\leq \lambda$ and $a_\ell \in \mathfrak{C}$ for $\ell = 0, 1$ such that $gtp(a_0/M; \mathfrak{C}) = gtp(a_1/M; \mathfrak{C})$, then there is $f \in Aut_M \mathfrak{C}$ such that $f(a) = b$.*

Proof: We use the notion of a special model. A model $\mathfrak{C}$ is *α-special* for α limit iff it has a resolution $\{M_i \mid i < \alpha\}$ such that M_{i+1} is $\|M_i\|^+$-Galois saturated. We can always build these under amalgamation by Proposition 3.16. Fix λ. With joint embedding, build M_0 that contains a copy of every model in $\mathcal{K}_{\mathrm{LS}(\mathcal{K})}$. Build a λ^+-special model $\mathfrak{C}$ that is witnessed by $\{M_i \mid i < \lambda^+\}$ starting with M_0. This model satisfies (1) and (2) above; we use cofinality arguments here. Note that $\|\mathfrak{C}\|$ is potentially around $\beth_{\lambda^+}$, but this is fine.

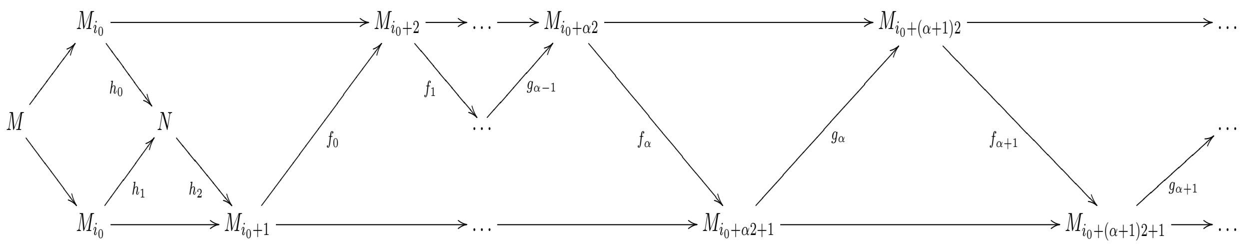
For (3), suppose that $M \prec \mathfrak{C}$ of size $\leq \lambda$ and $a_\ell \in \mathfrak{C}$ for $\ell = 0, 1$ such that $\mathrm{gtp}(a_0/M; \mathfrak{C}) = \mathrm{gtp}(a_1/M; \mathfrak{C})$. Then there is $i_0 < \lambda^+$ that contains a_0, a_1, M. By the definition of Galois type, there is $N \in \mathcal{K}_{\|M_{i_0}\|}$ and $h_\ell : M_{i_0} \to N$ such that $h_0(a_0) = h_1(a_1)$ and $h_0 \restriction M = h_1 \restriction M$. Now we build increasing, continuous $\{f_\alpha, g_\alpha \mid \alpha < \lambda^+\}$ by induction on α such that

(1) $f_\alpha : M_{i_0+\alpha 2} \to M_{i_0+\alpha 2+1}$ and $g_\alpha : M_{i_0+\alpha 2+1} \to M_{i_0+(\alpha+1)2}$;
(2) $f_\alpha \subset g_\alpha^{-1} \subset f_{\alpha+1}$; and
(3) there is $h_2 : N \to M_{i_0+1}$ such that $f_0 = h_2 \circ h_0$ amd $h_2 \circ h_1 = \mathrm{id}_{M_{i_0}}$.

We can do this by the fact that $M_{i_0+\beta+1}$ is $\|M_{i_0+\beta}\|^+$-model homogeneous. Then,

$$f = \cup_{\alpha<\lambda^+} f_\alpha = \cup_{\alpha<\lambda^+} g_\alpha^{-1}$$

is the desired automorphism. †



Exercise 3.22. *Prove the converse of Theorem 3.21.*

3.3. **Shelah's Presentation Theorem and the cardinal $\beth_{(2^\lambda)^+}$.** We've made reference to the following fact several times now.

Theorem 3.23. *Let $\mathcal{K}$ be an AEC with $LS(\mathcal{K}) = \kappa$. If, for every $\alpha < (2^\kappa)^+$, $\mathcal{K}_{\geq \beth_\alpha}$ is nonempty, then $\mathcal{K}$ has arbitrarily large models.*

We call this the *Hanf number (for existence)* of $\mathcal{K}$. We discuss Hanf number's in general more after the proof of Theorem 3.23. Theorem 3.23 gives an upper bound of this Hanf number, while the following example of Morley gives a lower bound.

Example 3.24 ([Mor65b])**.** *Suppose that α is definable in $\mathbb{L}_{\lambda^+,\omega}(\tau)$. Then there is $\psi \in \mathbb{L}_{\lambda^+,\omega}(\tau \cup \{E, P, <, r\})$ that has models of size $\beth_\alpha$ but no larger. Set ψ to be the conjunction of the following:*

$$\text{“}(P,<) \text{ has order type } \alpha\text{”} \wedge \text{“}E \text{ is extensional”} \wedge$$
$$\text{“}r \text{ is a unary function with range } P \text{ such that } xEy \text{ implies } r(x) < r(y)$$

Then models of ψ are isomorphic to substructures of $(V_\alpha, \in, \alpha, \in \mathit{rank})$, which is of size $\beth_\alpha$.

Note that at least every $\alpha < \lambda^+$ is definable in $\mathbb{L}_{\lambda^+,\omega}$. If λ is strong limit of cofinality ω, this is the upper bound. However, for general λ, the best known upper bound is $(2^\lambda)^+$. See the discussion after XXX for more.

The proof of this uses two powerful ingredients: Shelah's Presentation Theorem and Ehrenfuecht-Mostowski models.

Theorem 3.25 (Shelah's Presentation Theorem, [She, Conclusion 1.12])**.** *Let $\mathcal{K}$ be an AEC with $LS(\mathcal{K}) = \kappa$ in the language τ. Then there is a language $\tau_1 = \{F_i^n \mid n < \omega, i < LS(\mathcal{K})\} \cup \tau$ and a set of quantifier-free τ_1-types Γ such that*

1. *$M \in \mathcal{K}$ iff M is a τ-structure that has an expansion to a τ_1-structure M^* that omits Γ.*
2. *If M^* is a τ-structure that omits Γ and $N \in \mathcal{K}$, then $M^* \upharpoonright \tau \prec_{\mathcal{K}} N$ iff N has an expansion N^* to τ_1 that omits Γ so $M^* \subset N^*$.*
   In particular,
   $$\prec_{\mathcal{K}} = \left\{(M_1, M_2) \in \mathcal{K}^2 \mid M_\ell \text{ has an expansion } M_\ell^* \text{ that omits } \Gamma \text{ so } M_1^* \subset M_2^*\right\}$$

Moreover, these expansions are essentially Skolemizations, although I'm not quite sure how to formalize this. To prove this, we need the following generalization of the chain axioms and resolutions.

Exercise 3.26. *Any category closed under ordinal directed colimits is closed under all directed colimits. (See, for instance, [Grä68, Theorem 21.5].)*

Exercise 3.27 ([She09b, Lemma II.1.23])**.** *Given any $M \in \mathcal{K}$, we can find a directed system $\{M_{\mathbf{a}} \in \mathcal{K}_{LS(\mathcal{K})} \mid \mathbf{a} \in [M]^{<\omega}\}$ such that*

- *$\mathbf{a} \in M_{\mathbf{a}}$; and*
- *$\mathbf{a} \subset \mathbf{b} \in [M]^{<\omega}$ implies $M_{\mathbf{a}} \prec M_{\mathbf{b}}$.*

Moreover, $M \prec N$ iff we can find decompositions $\{M_i \mid i \in I\}$ of M and $\{N_i \mid i \in I\}$ of N such that for every $i \in I$, there is $j_i \in J$ such that $M_i \prec N_{j_i}$.

Proof of Theorem 3.25: Fix a τ_1-structure M^* and $a_1, \dots, a_n \in M^*$. Set $\langle a_1, \dots, a_n\rangle^{M^*}$ to be the (almost) τ_1-substructure of M^* with universe

$$\left\{(F_i^n)^{M^*}(a_1, \dots, a_n) \mid i < \mathrm{LS}(\mathcal{K})\right\}$$

Note that we are only closing under the n-ary functions, e. g., it's not guaranteed that $\left(F_i^{n-1}\right)^{M^*}(a_2, \dots, a_n) \in \langle a_1, \dots, a_n\rangle^{M^*}$. This is also the meaning of an "almost" substructure: we don't know that it's closed under all the functions of τ_1.

Note that the isomorphism type of $\langle a_1, \dots, a_n\rangle^{M^*}$ is entirely determined by the syntactic quantifier-free type

$$tp_{\tau_1}^{qf}(a_1, \dots, a_n/\emptyset; M^*)$$

This determines

- if $\langle a_1, \dots, a_n\rangle^{M^*}$ is a τ_1-structure;
- if $\langle a_1, \dots, a_n\rangle^{M^*} \upharpoonright \tau \in \mathcal{K}$; and
- given a subsequence $a_{i_1}, \dots, a_{i_m}$, if $\langle a_{i_1}, \dots, a_{i_m}\rangle^{M^*} \upharpoonright \tau \prec_{\mathcal{K}} \langle a_1, \dots, a_n\rangle^{M^*} \upharpoonright \tau$.

Now, we define our types

$$\begin{aligned}\Gamma = \{tp_{\tau_1}^{qf}(a_1, \dots, a_n/\emptyset; M^*) \quad | \quad & \langle a_1, \dots, a_n\rangle^{M^*} \upharpoonright \tau \notin \mathcal{K}; \text{ or} \\ & \text{there is a subsequence } a_{i_1}, \dots, a_{i_m} \text{ such that} \\ & \langle a_{i_1}, \dots, a_{i_m}\rangle^{M^*} \upharpoonright \tau \not\prec_{\mathcal{K}} \langle a_1, \dots, a_n\rangle^{M^*} \upharpoonright \tau\}\end{aligned}$$

Now we prove (1). First, let $M \in \mathcal{K}$. By Exercise 3.27, we can find a decomposition $\{M_{\mathbf{a}} \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})} \mid \mathbf{a} \in [M]^{<\omega}\}$ of M. Expand M to a τ_1-structure by setting $\{(F_i^n)^{M^*}(a_1, \dots, a_n) \mid i < \mathrm{LS}(\mathcal{K})\}$ to be an enumeration of $M_{\mathbf{a}}$. Then $\langle a_1, \dots, a_n\rangle^{M^*} = M_{\mathbf{a}}$, so M^* omits Γ.

Second, let M be a τ-structure with an expansion M^* that omits Γ. Define $M_{\mathbf{a}}$ to be the τ-substructure of M that has universe $\{(F_i^n)^{M^*}(a_1, \dots, a_n) \mid i < \mathrm{LS}(\mathcal{K})\}$. By the omission of Γ, this forms a directed system, so $\cup_{\mathbf{a}\in[M]^{<\omega}} M_{\mathbf{a}} \in \mathcal{K}$ and this union is M.

Now we prove (2). The proof is essentially the same as above with the following changes: the expansion of M^* induces a decomposition $\{M_{\mathbf{a}} \mid \mathbf{a} \in [M]^{<\omega}\}$ of M and an enumeration of each $M_{\mathbf{a}}$. Find a decomposition $\{N_{\mathbf{a}} \mid \mathbf{a} \in [N]^{<\omega}\}$ of N to use such that $M_{\mathbf{a}} = N_{\mathbf{a}}$ whenever $\mathbf{a} \in [M]^{<\omega}$ and use the already existing enumeration of it. †

A gloss of this theorem is that every AEC has an "abstract Skolemization" to an $\mathbb{L}_{\mathrm{LS}(\mathcal{K})^+,\omega}$-theory. Since Γ has an element for each bad τ_1-structure, it seems like Γ will almost always have the maximum size $(2^{\mathrm{LS}(\mathcal{K})})^+$. Thus, this $\mathbb{L}_{\mathrm{LS}(\mathcal{K})^+,\omega}$-theory is a $\mathbb{L}_{(2^{\mathrm{LS}(\mathcal{K})})^+,\omega}$-sentence. So to prove the Hanf number, we use the proof of the Hanf numbers for $\mathbb{L}_{\infty,\omega}$.

This uses the technology of indiscernibles and Ehrenfuecht-Mostowski models.

Definition 3.28. *Fix a language τ and a fragment of $\mathcal{F} \subset \mathbb{L}_{\lambda,\omega}(\tau)$.*

(1) *A sequence $\{\mathbf{a}_i \in {}^nM \mid i \in I\}$ are* order $\mathcal{F}$-indiscernible *iff for every $\phi(x_1, \dots, x_n) \in \mathcal{F}$ and increasing sequences $i_1 < \dots i_n$, $j_1 < \dots < j_n$ from I, we have*

$$M \vDash \phi(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}) \iff M \vDash \phi(\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_n})$$

(2) *A* blueprint Φ *is a collection $\{p_n \mid n < \omega\}$ so that $p_n(x_1, \dots, x_n)$ is a complete quantifier-free type in a fixed language such that*
   (a) *"$x_1 \neq x_2$" $\in p_2$; and*
   (b) *for any sequence $k_1 < \dots < k_m \leq n$, we have $p_n^{k_1,\dots,k_n}$.*

   $\tau(\Phi)$ is the fixed language.

(3) *Given a linear order I, a blueprint Φ, and $\tau \subset \tau_1$, the model $EM_\tau(I, \Phi)$ is the τ-structure formed as follows:*
   - *$X := \{\sigma(i_1, \dots, i_n) \mid i_1, \dots, i_n \in I, \sigma$ is a composition of functions in $\tau_1\}$*
   - *Given $\sigma_0(i_1, \dots, i_n), \sigma_1(i_1, \dots, i_n) \in X$ with $i_1 < \dots < i_n$ (and possibly increasing the free variables of the terms), set $\sigma_0(i_1, \dots, i_n) \sim \sigma_1(i_1, \dots, i_n)$ iff "$\sigma_0(x_1, \dots, x_n) = \sigma_1(x_1, \dots, x_n)$" $\in p_n$.*
   - *For n-ary $f \in \tau$, set*

   $$f^+([\sigma(i_1, \dots, i_n)]) = [f \circ \sigma(i_1, \dots, i_n)]$$

   - *For n-ary $R \in \tau$, set*

   $$[\sigma(i_1, \dots, i_n)] \in R^+ \iff \text{“}R(\sigma(x_1, \dots, x_n))\text{''} \in p_n$$

   - *$EM(I, \Phi) = (X/\sim, f^+, R^+)_{f,R\in\tau_1}$*
   - *$EM_\tau(I, \Phi) = EM(I, \Phi) \upharpoonright \tau$*

(4) *For an AEC $\mathcal{K}$ (or more general classes), we say that a blueprint Φ is* proper for $\mathcal{K}$ for linear orders *iff $\tau(\mathcal{K}) \subset \tau(\Phi)$; for every linear order I, $EM_{\tau(\mathcal{K})}(I, \Phi) \in \mathcal{K}$; and if $I \subset J$, then $EM_\tau(I, \Phi) \prec_{\mathcal{K}} EM_\tau(J, \Phi)$.*

(5) *$\Upsilon[\mathcal{K}]$ is the class of all blueprints that are proper for linear orders and $\Upsilon_\kappa[\mathcal{K}] := \{\Phi \in \Upsilon[\mathcal{K}] \mid |\tau(\Phi)| \leq \kappa\}$.*

Remark 3.29. *We could try to be much more general here. Two generalizations come to mind:*

(1) *Looking at more general logics. Shelah's Presentation Theorem 3.25 says that any logic that forms an AEC has a Skolemization to $\mathbb{L}_{\lambda,\omega}$, so these logics can be treated similarly. If*

we try and extend it to a logic with an infinitary Skolemization, this poses problems. The theory above generalizes, but actually finding indiscernibles will use Ramsey principles and infinite arity Ramsey principles run afoul of Axiom of Choice (and we're using choice throughout).

(2) *Looking at objects other than linear orders. Again, the statements above generalize, but finding indiscernibles based on objects other than linear orders is harder. In first-order, where Ramsey's Theorem suffices, there is a general theory of Ramsey classes (see [GHS] for classification theory applications). However, we will use Erdős-Rado, which has no analogue in this context.*

Theorem 3.30 (Erdős-Rado, [ER56])**.** *For any cardinal κ and $n < \omega$,*

$$\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$$

The following is a model-theoretic proof due to Steve Simpson (following [Kei71, Theorem 20]).

Proof: For $n = 0$, this is the pigeonhole principle.

Fix $n > 0$ and assume $\beth_{n-1}(\kappa)^+ \to (\kappa^+)^n_\kappa$. Let $c : \beth_n(\kappa)^+ \to \kappa$ be a coloring. Create the structure

$$M = \left(\beth_n(\kappa)^+, R, \alpha\right)_{\alpha<\kappa}$$

where c_α is a constant symbol interpreted as α and R is an $n+2$-ary relation so $(\alpha_1, \dots, \alpha_{n+1}, \alpha) \in R$ iff $c(\alpha_1, \dots, \alpha_{n+1}) = \alpha$. The following is true (see Exercise 3.32).

Claim 3.31. *There is $N \prec M$ of size $\beth_n(\kappa)$ containing κ that is $\beth_{n-1}(\kappa)^+$-saturated relative to M; this means any type over a subset of N of size $< \beth_{n-1}(\kappa)^+$* that is realized in M *is realized in N.*

Since $\|N\| < \|M\|$, there is $x \in M - N$. We build $\{y_\alpha \in N \mid \alpha < \beth_{n-1}(\kappa)^+\}$ by induction so that

$$tp\,(y_\alpha/\{y_\beta \mid \beta < \alpha\}; N) = tp\,(x/\{y_\beta \mid \beta < \alpha\}; M)$$

This is possible by the relative saturation. Now define a coloring

$$d : [\{y_\alpha \mid \alpha < \beth_{n-1}(\kappa)^+\}]^n \to \kappa$$

by $d(y_{\alpha_0}, \dots, y_{\alpha_{n-1}}) = c(y_{\alpha_0}, \dots, y_{\alpha_{n-1}}, x)$. By induction, there is $Y \subset \{y_\alpha \mid \alpha < \beth_{n-1}(\kappa)^+\}$ that is homogeneous for d with color α. Then Y is homogeneous for c with color α. †

Exercise 3.32. *Prove Claim 3.31. Hint: imitate the proof of Proposition 3.16.*

The following is Morley's Omitting Types Theorem. We will see a more powerful version Shelah's Omitting Types Theorem 5.39 later.

Theorem 3.33 (Morley's Omitting Types Theorem, [Mor65b])**.** *Let τ be a language, T a first-order theory, Γ a collection of τ-types and set $\mu = (2^{|\tau|})^+$ and $\mathcal{K}$ to be the class of models of T omitting Γ. Suppose that for every $\alpha < \mu$, $\mathcal{K}_{\geq \beth_\alpha} \neq \emptyset$. Then there is $\Phi \in \Upsilon_{|\tau|}[\mathcal{K}]$. In particular, for all λ, $\mathcal{K}_{\geq\lambda} \neq \emptyset$.*

This theorem is really about finding indiscernibles that are finitely appearing in these models.

Proof: WLOG T has Skolem functions (this doesn't change the size of the language) and Γ are quantifier-free types. For each $\alpha < \mu$, find $M_\alpha \in \mathcal{K}_{\beth_\alpha}$.

We will build sequences of injective functions f^n_α with domain $\beth_\alpha$ and range $M_{\beta_n(\alpha)}$ with $\alpha \leq \beta_n(\alpha) < \mu$ by induction on $n < \omega$ such that

(1) for $\alpha < \mu$ and $n < \omega$,

$$tp^{qf}_\tau \left(f^n_\alpha(i_1), \dots, f^n_\alpha(i_n)/\emptyset; M_{\beta_n(\alpha)} \right)$$

is constant for every $i_1 < \cdots < i_n < \beth_\alpha$.

(2) for $\alpha < \mu$ and $m < n$, there is $\alpha < \beta < (2^\mu)^+$ such that $\{f^n_\alpha(i) \mid i < \beth_\alpha\}$ is an increasing subset of $\{f^m_\beta(i) \mid i < \beth_\beta\}$.

This is enough: Define Φ by setting $p_n = tp^{qf}_\tau \left(f^n_\alpha(i_1), \dots, f^n_\alpha(i_n)/\emptyset; M_{\beta_n(\alpha)} \right)$ for any $\alpha < \mu$ and $i_1 < \cdots < i_n < \beth_\alpha$. By construction, this is a blueprint and $|\tau(\Phi)| = |\tau|$. Now we wish to show it is in $\Upsilon[\mathcal{K}]$.

Fix a linear order I and set $M = EM_\tau(I, \Phi)$. If some $p(x_1, \dots, x_n) \in \Gamma$ is not omitted, then there is $i_1 < \cdots < i_m \in I$ and terms $\sigma_1, \dots, \sigma_n$ such that $\sigma_1(i_1, \dots, i_m), \dots, \sigma_n(i_1, \dots, i_m)$ realizes p. Since everything is quantifier-free, the fact that p is realized is already witnessed by $tp^{qf}(i_1, \dots, i_m/\emptyset; EM(I, \Phi)) = p_m$. However, we know that p_m is realized in (many) models that omit p, so this is a contradiction.

Construction: $\mathbf{n = 0}$: Set $\beta_0(\alpha) = \alpha$ and let f^0_α enumerate M_α; we could start with some other large set to ensure that the indiscernibles had some property.

$\mathbf{n + 1}$: Fix $\alpha < (2^\mu)^+$. Color $[\beth_{\alpha+\omega}]^{n+1}$ with

$$tp^{qf}_\tau \left(f^n_{\alpha+\omega}(i_1), \dots, f^n_{\alpha+\omega}(i_{n+1})/\emptyset; M_{\beta_n(\alpha+\omega)} \right)$$

By hypothesis, this coloring is monochromatic on the n-tuples. By the Erdős-Rado Theorem 3.30, $\beth_{\alpha+\omega} \to (\beth_\alpha)^{n+1}_{2^\lambda}$. Then, we can find $Y^{n+1}_\alpha \subset \beth_{\alpha+\omega}$ of size $\beth_\alpha$ so that this coloring is constant. Define $\hat{f}^{n+1}_\alpha : \beth_\alpha \to M_{\beta_n(\alpha+\omega)}$ by

$$\hat{f}^{n+1}_\alpha(i) = f^n_{\alpha+\omega}(j)$$

where $j \in Y^{n+1}_\alpha$ is the unique member so $\operatorname{otp}(j \cap Y^{n+1}_\alpha) = i$.

However, there is no guarantee that this constant color is the same across all α's. So, for $\alpha < \mu = (2^{|\tau|})^+$, let p^α be the constant color on Y^{n+1}_α. There are only $2^{|\tau|}$-many options, so the pigeonhole principle implies there is $X^{n+1} \subset \mu$ of size μ such that the colors are constant. Finish this step by setting

$$\begin{aligned} f^{n+1}_\alpha &= \hat{f}^{n+1}_\gamma \\ \beta_{n+1}(\alpha) &= \beta_n(\gamma + \omega) \end{aligned}$$

where $\gamma \in X^{n+1}$ is the unique element so $\operatorname{otp}(\gamma \cap X^{n+1}) = \alpha$. †

Remark 3.34. *For later (Theorem 4.29), convince yourself that the following statement holds:*

If for each $\alpha < (2^{|\tau|})^+$, M_α is a τ-structure; $(I_\alpha, <_\alpha)$ is a linear order of size $\beth_\alpha$; and $f_\alpha : I_\alpha \to M_\alpha$ is an injection, then there is a blueprint $\Phi = \{p_n \mid n < \omega\}$ with $\tau(\Phi) = \tau$ such that for every $n < \omega$, there are cofinally many $\alpha < (2^{|\tau|})^+$ and (many) $i^n_1 <_\alpha \cdots <_\alpha i^n_n \in I_\alpha$ such that

$$p_n = tp_{qf} \left(f_\alpha(i^n_1), \dots, f_\alpha(i^n_n)/\emptyset; M_\alpha \right)$$

The proof is the same except the $n = 0$ step starts with the given functions and we don't need to worry about reordering.

Proof of 3.23: Let $\mathcal{K}$ be an AEC with models of size cofinal in $\beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$. Use Shelah's Presentation Theorem 3.25 to expand all of these models to τ_1-structures that omit Γ. By Morley's Omitting Types Theorem 3.33, there is $\Phi \in \Upsilon_{\mathrm{LS}(\mathcal{K})}[\mathcal{K}]$. Then $\|EM_\tau(I, \Phi)\| = |I| + \mathrm{LS}(K)$. †

There is a different method of proof that yields a (potentially) stronger result. Define what is sometimes called the pinning down ordinal:

$$\delta(\lambda,\kappa) = \min\{\delta \mid \begin{array}{l}\text{for all first order theories } T \text{ of size} \leq \lambda \text{ with } <, P \in \tau(T) \text{ and for all } \Gamma \text{ of}\\ \text{size} \leq \kappa \text{ many types, if there is } M \vDash T \text{ omitting } \Gamma \text{ such that}\\ \text{otp}(P^M, <^M) \geq \delta \text{, then there is } N \vDash T \text{ omitting } \Gamma \text{ such that } (P^N, <^N) \text{ is not}\\ \text{well-ordered.}\}\end{array}$$

Example 3.24 shows that the Hanf number is at least $\beth_{\delta(\mathrm{LS}(\mathcal{K}), 2^{\mathrm{LS}(\mathcal{K})})}$ and further results show that $\delta(\kappa, \lambda) \leq \delta(\lambda, \lambda) = \delta(\lambda, 1) \leq (2^\lambda)^+$, giving us our result. One can (which should be interpreted as "Shelah has") argued using ill-founded models of set-theory that the Hanf number is exactly $\beth_{\delta(\lambda,\lambda)}$.

We've been discussing the Hanf number for existence, but you can talk about the Hanf number for any property P: Given a class $\mathfrak{K}$ of classes of structures and a property P,

$$H^P_{\mathfrak{K}} = \min\{\lambda \mid \forall \mathcal{K} \in \mathfrak{K}, \text{ if some } M \in \mathcal{K}_{\geq\lambda} \text{ satisfies } P, \text{ then cofinally many } M \in \mathcal{K} \text{ satisfy } P\}$$

Write H^P_λ for $\mathfrak{K}$ being all AECs with $\mathrm{LS}(\mathcal{K}) \leq \lambda$.

There is a quicker proof than the above that there is a Hanf number for existence.

Proposition 3.35.

(1) *For each λ, there are set-many AECs with Löwenheim-Skolem-Tarski number $\leq \lambda$.*
(2) *There is a Hanf number for any property for each λ.*

Proof: Consider the map for AECs with $\mathrm{LS}(\mathcal{K}) \leq \lambda$ $\mathcal{K} \mapsto \mathcal{K}^{card}_\lambda$, which is the collection of models of size λ whose universe is a subset of λ. There are clearly set many things in the image of this map (in fact $\leq 2^{2^\lambda}$). We claim that this map is almost injective in the sense that if $(\mathcal{K}^1)^{card}_\lambda = (\mathcal{K}^2)^{card}_\lambda$, then $\mathcal{K}^1_{\geq\lambda} = \mathcal{K}^2_{\geq\lambda}$. This is essentially the content of [She09b, Lemma II.1.23] that is cited for Exercise 3.27: given $\mathcal{K}^{card}_\lambda$, close it under isomorphisms and then directed colimits; this will generate $\mathcal{K}_{\geq\lambda}$. Thus, there are set-many AECs with $\mathrm{LS}(\mathcal{K}) \leq \lambda$.

Fix a property P and set $spec_P(\mathcal{K}) = \{\mu \mid \mathcal{K}_\mu \text{ satisfies } P\}$. Define $\mu^P_{\mathcal{K}} = \sup spec_P(\mathcal{K})$ and

$$H^P_\lambda = \sup\left\{\mu^P_{\mathcal{K}} \mid \mathrm{LS}(\mathcal{K}) \leq \lambda, \mu_{\mathcal{K}} < \infty\right\}$$

Then if $\mathcal{K}$ has $\mathrm{LS}(\mathcal{K}) = \lambda$ and $\mathcal{K}_{H^P_\lambda}$ satisfies P, then cofinally many $\mathcal{K}_\lambda$ satisfy P. †

So Hanf numbers for exist, but we have no bound (the proof above uses replacement). Note that if P is downward closed, we can conclude P on a tail. See [BB17, Section 4] for Hanf numbers in general.

Here are some further applications of EM models:

1) **Stability below a categoricity cardinal:** This essence of this proof in the first-order case (I think) goes back to Morley.

Definition 3.36 (Galois stability)**.** *$\mathcal{K}$ is λ-Galois stable iff for every $M \in \mathcal{K}_\lambda$, $|gS(M)| = \lambda$.*

Theorem 3.37. *Suppose $\Phi \in \Upsilon_{LS(\mathcal{K})}[\mathcal{K}]$. If $\mathcal{K}$ is categorical in λ and $\mathcal{K}_{\leq\lambda}$ has amalgamation and joint embedding, then $\mathcal{K}$ is μ-Galois stable for every $\mu < \lambda$.*

Proof: First observe that it is enough to find a particular model in $\mathcal{K}_\lambda$ such that every μ-sized submodel has μ-many types over it. Then, by joint embedding and categoricity, this gives μ-stability.

Set $N := EM_\tau(\lambda, \Phi)$. Let $M \prec N$ of size μ. Then there is $Y \subset \lambda$ of order type $\alpha < \mu^+$ such that $M \prec EM_\tau(Y, \Phi)$. Set $N[\mathrm{gS}(M)] = \{p \in \mathrm{gS}(M) \mid N \text{ realizes } p\}$; we want to show

$|N[\text{gS}(M)]| \leq \mu$. For each $p \in N[\text{gS}(M)]$, find $a_p \in N$ realizing it. By the definition of EM-models, there is is $\tau(\Phi)$-term σ_p; $\alpha_1^p < \cdots < \alpha_{n_p}^p \in Y$; and $\beta_1^p < \cdots < \beta_{m_p}^p \in \lambda - Y$ such that

$$a_p = \sigma_p(\alpha_1^p, \dots, \alpha_{n_p}^p, \beta_1^p, \dots, \beta_{m_p}^p)$$

Look at the map

$$p \in N[\text{gS}(M)] \mapsto \Big(\sigma_p, n_p, m_p, tp_{qf}(\alpha_1^p, \dots, \alpha_{n_p}^p, \beta_1^p, \dots, \beta_{m_p}^p / Y; \lambda)\Big)$$

First, we claim that the image of this map is μ-sized. $\tau(\Phi) \leq \text{LS}(\mathcal{K}) \leq \mu$ and there are only μ many choices for the type (first pick an ordinal in $[\alpha, \alpha + \omega)$ and then pick the finitely many elements that are the β sequence).

Second, we claim this map is an injection. If $\sigma = \sigma_p = \sigma_q$, $n = n_p = n_q$, $m = m_p = m_q$, and $f = \text{id}_Y \cup \{(\beta_\ell^p, \beta_\ell^q) \mid \ell \leq m\}$ is order-preserving, then f lifts to an isomorphism

$$f^* : EM\left(Y \cup \{\beta_\ell^p \mid \ell \leq m\}, \Phi\right) \cong EM\left(Y \cup \{\beta_\ell^q \mid \ell \leq m\}, \Phi\right)$$

(see Exercise 3.39 for details). Restricting to τ, this gives us the diagram

$$\begin{tikzcd}
 & EM_\tau\left(Y \cup \{\beta_\ell^p \mid \ell \leq m\}, \Phi\right) \arrow[r, "f^* \restriction \tau", "\cong"'] & EM_\tau\left(Y \cup \{\beta_\ell^q \mid \ell \leq m\}, \Phi\right) \\
 & EM_\tau\left(Y, \Phi\right) \arrow[u] \arrow[r] & EM_\tau\left(Y \cup \{\beta_\ell^q \mid \ell \leq m\}, \Phi\right) \arrow[u] \\
M \arrow[ur] & &
\end{tikzcd}$$

and, since f^* is a $\tau(\Phi)$-isomorphism,

$$\begin{aligned}
f^*(a_p) &= f^*\Big(\sigma(\alpha_1^p, \dots, \alpha_{n_p}^p, \beta_1^p, \dots, \beta_{m_p}^p)\Big) = \sigma\Big(f(\alpha_1^p), \dots, f(\alpha_{n_p}^p), f(\beta_1^p), \dots, f(\beta_{m_p}^p)\Big) \\
&= \sigma(\alpha_1^q, \dots, \alpha_{n_q}^q, \beta_1^q, \dots, \beta_{m_q}^q) = a_q
\end{aligned}$$

Thus $p = q$. †

2) **Galois indiscernibles:** We can generalize Definition 3.28.(1) to AECs.

Definition 3.38. *$\langle \mathbf{a}_i \mid i \in I\rangle \subset M$ are called* Galois indiscernibles over $N \prec M$ *iff for every $i_1 < \cdots < i_n, j_1 < \cdots < j_n \in I$,*

$$gtp(\mathbf{a}_{i_1}, \dots, i_n / N; M) = gtp(\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_n} / N; M)$$

Exercise 3.39. *If $\Phi \in \Upsilon[\mathcal{K}]$, then $I \subset EM_\tau(I, \Phi)$ are Galois indiscernibles. Moreover, these are* strict *indiscernibles, which means any $f : I \to I$ can be extended to a strong embedding.*

4. Tameness

We're now ready to define tameness!! Tameness is one of many type locality properties (Definition 4.1) that one can define. I list a bunch here for completeness, but we're most interested in tameness, in particular, in $< \kappa$-tameness, although we will care about (μ, λ)-weak tameness for a bit (see Theorem 4.25).

Definition 4.1. *Let $\mathcal{K}$ be an AEC.*

(1) *$\mathcal{K}$ is* $(< \kappa, \lambda)$-tame for β-types *iff for every $M \in \mathcal{K}_\lambda$ and $p \neq q \in gS^\beta(M)$, there is $M_0 \prec M$ such that $\|M\| < \kappa$ and $p \restriction M \neq q \restriction M$.*

(2) $\mathcal{K}$ *is* (κ, λ)-local for β-types *iff for every* $M \in \mathcal{K}_\lambda$*; continuous, increasing* $\langle M_i \mid i < \kappa\rangle$ *with* $M = \cup_{i<\kappa} M_i$*; and* $p \neq q \in gS^\beta(M)$*, there is* $i_0 < \kappa$ *such that* $p \upharpoonright M_{i_0} \neq q \upharpoonright M_{i_0}$.
(3) $\mathcal{K}$ *is* (κ, λ)-compact for β-types *iff for every* $M \in \mathcal{K}_\lambda$*; continuous, increasing* $\langle M_i \mid i < \kappa\rangle$ *with* $M = \cup_{i<\kappa} M_i$*; and increasing* $\langle p_i \in gS^\beta(M_i) \mid i < \kappa\rangle$*, there is* $p \in gS^\beta(M)$ *such that* $p_i \leq p$ *for all* $i < \kappa$.
(4) $\mathcal{K}$ *is* $(< \kappa, \lambda)$-type short over μ-sized domains *iff for every* $M \in \mathcal{K}_\mu$ *and* $p \neq q \in gS^\lambda(M)$*, there is* $X \in \mathcal{P}_\kappa \lambda$ *such that* $p^X \neq q^X$.
(5) *For any of* P *being tame, local, compact, or type short, we introduce the following variations:*
  (a) *Omitting for* β*-types means* $\beta = 1$.
  (b) (κ, λ)*-*P *means* $(< \kappa^+, \lambda)$*-*P.
  (c) $< \kappa$*-*P *or* $(< \kappa, \infty)$*-*P *means* $(< \kappa, \lambda)$*-*P *for all* λ.
  (d) P *means* $< \kappa$*-*P *for some* P.
  (e) *Weakly* $(< \kappa, \lambda)$*-*P *restricts* M *to be a Galois saturated model (and is only used when such models are plentiful).*

Additionally, we allow tameness with $\kappa < LS(\mathcal{K})$ *by considering types over sets.*

There are several obvious relations between these properties, which we list below. The items regarding compactness are less trivial (see [**?**, **?**,]).

Proposition 4.2.
(1) $(cf\, \lambda, \lambda)$*-locality implies* $(< \lambda, \lambda)$*-tameness.*
(2) $< cf\, \kappa$*-tameness implies* κ*-locality.*
(3) $(< \kappa, \mu)$*-type shortness over the empty set implies* (κ, μ)*-tameness for* $\leq \mu$*-types.*
(4) *Every AEC with amalgamation is* ω*-compact for all lengths.*
(5) *If* $\mathcal{K}$ *is* κ_**-local for every* $\kappa_* < \kappa$ *and* $\mathcal{K}$ *has amalgamation, then* $\mathcal{K}$ *is* κ_**-compact for every* $\kappa_* \leq \kappa$.

Proof: We blend a proof of (4) and (5). Suppose $\langle M_n \mid n \leq \omega\rangle$ is increasing and $p_n \in \mathrm{gS}(M_n)$ such that $p_n = p_{n+1} \upharpoonright M_n$. Write $p_n = \mathrm{gtp}(a_n/M_n; N_n)$. We are going to build a coherent system by induction; that is, increasing, continuous $\langle N_n^*, f_n : N_n \to N_n^* \mid n < \omega\rangle$ such that
(1) $f_0 = id$ and $N_0^* = N_0$;
(2) $f_n(a_n) = a_0$; and
(3) $f_n \upharpoonright M_n = f_{n+1} \upharpoonright M_{n+1}$.

We can build this by induction: $f_n^{-1}\left(\mathrm{gtp}(a_0/f_n(M_n); N_n^*)\right) = p_n$ by construction and this is $p_{n+1} \upharpoonright M_n$. So find witnesses to this Galois type equality.

In the end, we have $f_\omega := \cup_{n<\omega} f_n \upharpoonright M_n : N_\omega^* \to M_\omega$ where $N_\omega^* = \cup_{n<\omega} N_n^*$. We can do a renaming to find $N_\omega \succ M_\omega$ and $g : N_\omega^* \cong N_\omega$ such that $g \circ f_\omega = \mathrm{id}$. Then set $q = \mathrm{gtp}\left(g(a_0)/M_\omega; N_\omega\right)$. The following diagram shows that $p_n \leq q$:

$$\begin{tikzcd}
N_n \arrow[r, "f_n"] & N_n^* \arrow[r] & N_\omega^* \\
M_n \arrow[u] \arrow[r] & M_\omega \arrow[r] & N_\omega \arrow[u, "g^{-1}"']
\end{tikzcd}$$

This finishes (4). In the case of (5), the given locality ensures that $q = p_\omega$, and this construction can continue. †

A particularly nice version of this is being *fully* $< \kappa$*-tame and -type short*, where 'fully' indicates that it holds for all choices of β and μ. This means that any Galois type is determined by its 'small approximations,' that is, restrictions to domains and length of size $< \kappa$. Note that the tameness is included mostly for historical reasons (see [Bon14, BG17]) as Proposition 4.2.(3) shows it is implied by the type shortness. Then we can identify a Galois type p with the set of its small approximations. Vasey [Vas16c] has taken this further to show that, in such cases, Galois types can be identified with syntactic types in a functorial expansion.

Fix an AEC K in a language τ and a cardinal κ. From the proof of Proposition 4.2.(3), we can restrict to types over the empty set (we didn't actually prove this, but if one were to, this would become more clear).

- $\tau^{*\kappa} := \tau \cup \{R_q \mid q \in \mathrm{gS}^{<\kappa}_{\mathcal{K}}(\emptyset)\}$, where R_q is a $\ell(q)$-ary relation.
- For $M \in \mathcal{K}$, $M^{*\kappa}$ is the expansion of M to a $\tau^{*\kappa}$-structure by setting $R_q^{M^{*\kappa}} = \{\mathbf{a} \in M \mid \mathbf{a} \vDash p\}$.
- $\mathcal{K}^{*\kappa} = \{M^{*\kappa} \mid M \in \mathcal{K}\}$ and $M^{*\kappa} \prec_{\mathcal{K}^{*\kappa}} N^{*\kappa}$ iff $M^{*\kappa} \upharpoonright \tau \prec_{\mathcal{K}} N^{*\kappa} \upharpoonright \tau$.

Then for any Galois type $p = \mathrm{gtp}(\langle a_i \mid i < \lambda\rangle/\emptyset, M) \in \mathrm{gS}^{<\infty}_{\mathcal{K}}(\emptyset)$, we can define

$$p^{*\kappa} := tp^{\tau^{*\kappa}}_{qf}(\langle a_i \mid i < \lambda\rangle/\emptyset, M^{*\kappa}) = \left\{R_q(x_{i_\alpha} \mid \alpha < \ell(q)) \mid q \in \mathrm{gS}^{<\kappa}_{\mathcal{K}}(\emptyset), \langle a_{i_\alpha} \mid \alpha < \ell(q)\rangle \vDash q\right\}$$

Proposition 4.3 ([Vas16c, Theorem 3.16]). *The map $p \mapsto p^{*\kappa}$ is injective iff $\mathcal{K}$ is fully $< \kappa$-tame and -type short.*

4.1. **General ways to get tameness.** Tameness is a very nice property. The natural follow-up is how often and where it occurs. After it's earliest uses by Grossberg and VanDieren [GV06b, GV06c, GV06a] to, in particular, prove an upward categoricity transfer, some felt that it was very strong. Here we present three general ways of finding tameness: the existence of sufficiently complete ultraproducts, being in a universal class, or categoricity. My feeling is that these (and other results) indicate that tameness is quite natural.

4.1.1. *Sufficiently complete ultraproducts.* This section essentially uses three properties (and reports results of [Bon14] with later improvements):

- κ-complete ultraproducts have a Łoś' Theorem for $\mathbb{L}_{\kappa,\kappa}$ (e.g., [CK12, Theorem 4.2.11]);
- Shelah's Presentation Theorem 3.25 shows every AEC has an expansion to a $\mathbb{L}_{\mathrm{LS}(\mathcal{K})^+,\omega}$-axiomatizable class; and
- ultraproducts commute with reducts.

Combining these gives Łoś' Theorem for AECs [Bon14, Theorem 4.3] ([Bon14, Theorem 4.7] also has this name and extends the result from models to types). We lack a syntactic version of Łoś' Theorem precisely because we lack syntax for AECs. However, we can give a semantic version.

Theorem 4.4 (Łoś' Theorem for AECs, [Bon14, Theorem 4.3]). *If $\mathcal{K}$ is an AEC, then both $\mathcal{K}$ and $\prec_{\mathcal{K}}$ are closed under $LS(\mathcal{K})^+$-complete ultraproducts.*

Proof: It is enough to prove this for $\prec_{\mathcal{K}}$ as $\mathcal{K}$ is the collection of τ-structures M such that $M \prec_{\mathcal{K}} M$. Let U be an $\mathrm{LS}(\mathcal{K})^+$-complete[7] ultrafilter on I and $M_i \prec_{\mathcal{K}} N_i$ for $i \in I$. Let τ_1 and Γ_1 be the language and types from Shelah's Presentation Theorem 3.25. Then there are expansions $M_i^* \subset N_i^*$ that omit Γ_1. Omitting a type of size $\mathrm{LS}(\mathcal{K})$ is a sentence in$\mathbb{L}_{\mathrm{LS}(\mathcal{K})^+,\omega}$, so Łoś for this logic gives that $\prod M_i^*/U \subset \prod N_i^*/U$ both omit Γ_1. Finally, since ultraproducts commute with reducts

$$\prod M_i/U = \prod (M_i^* \upharpoonright \tau)/U = \left(\prod M_i^*/U\right) \upharpoonright \tau \prec_{\mathcal{K}} \left(\prod N_i^*/U\right) \upharpoonright \tau = \prod N_i/U$$

†

[7]The actual completeness of any ultrafilter is ω or a measurable.

Exercise 4.5. *Show that the ultrapower embedding is a strong embedding when the ultrafilter is $LS(\mathcal{K})^+$-complete.*

Ultraproducts give us compactness and tameness is a fragment of compactness, so perhaps it comes as no surprise that large cardinals give tameness (however, I think it is surprising that tameness gives large cardinals, see [BU17] or Theorem **??**)

Theorem 4.6 ([Bon14, Theorem 4.5])**.** *If κ is strongly compact and $\mathcal{K}$ is an AEC with $LS(\mathcal{K}) < \kappa$, then $\mathcal{K}$ is fully $< \kappa$-tame and -type short.*

Proof: We assume amalgamation for ease; see Exercise 4.7.

By Proposition 4.2.(3), it is enough to show the type shortness over the empty set. Suppose that $\mathbf{a}^\ell_\lambda := \langle a^\ell_i \mid i < \lambda\rangle \subset N_\ell$ for $\ell = 1, 2$ such that, for $X \in \mathcal{P}_\kappa\lambda$,

$$\mathrm{gtp}\left(\langle a^1_i \mid i \in X\rangle/\emptyset; N_1\right) = \mathrm{gtp}\left(\langle a^2_i \mid i \in X\rangle/\emptyset; N_2\right)$$

Write $\mathbf{a}^\ell_X$ for $\langle a^1_i \mid i \in X\rangle$. WLOG $\|N_1\| = \|N_2\| = \lambda$. There is N^X and $f^X_\ell : N_\ell \to N^X$ such that $f^X_1(\mathbf{a}^1_X) = f^X_2(\mathbf{a}^2_X)$.

Let U be a κ-complete, fine ultrafilter on $\mathcal{P}_\kappa\lambda$. Then, by Łoś' Theorem for AECs 4.4 implies that the following diagram is of strong embeddings

$$\begin{tikzcd}
& \prod N^X/U & \\
\prod N_1/U \arrow[ur, "\prod f_1^X/U"] & & \prod N_2/U \arrow[ul, "\prod f_2^X/U"'] \\
N_1 \arrow[u, "J"'] & & N_2 \arrow[u, "J"']
\end{tikzcd}$$

†

Exercise 4.7. *Show that the assumption of amalgamation is unnecessary in the theorem above.*

Exercise 4.8. *Use the methods of the above theorem to show:*

- *If κ is measurable, then $\mathcal{K}$ is κ-local [Bon14, Theorem 5.2].*
- *If κ is weakly compact, then $\mathcal{K}$ is $(< \kappa, \kappa)$-tame [Bon14, Theorem 6.4].*

We can use ultraproducts to get much more. The use of strongly compact cardinals in nonelementary classification theory goes back to Makkai and Shelah [MS90] that considered categoricity in $\mathbb{L}_{\kappa,\omega}$ with κ strongly compact. We show how to use strong compacts to prove amalgamation. Many of the results in this section can weaken '$\mathrm{LS}(\mathcal{K}) < \kappa$' to '$\mathrm{LS}(\mathcal{K}) < \kappa$ or $\mathcal{K}$ is $\mathbb{L}_{\kappa,\kappa}$-axiomatizable.'

Theorem 4.9.

(1) *If κ is strongly compact and $\mathcal{K}$ with $LS(\mathcal{K}) < \kappa$ is categorical in $\mu^{<\kappa} = \mu$, then $\mathcal{K}_{\geq\kappa}$ has amalgamation [Bon14, Proposition 7.3].*

(2) *Strongly compact cardinals are Hanf numbers for amalgamation [BB17, Theorem 1.0.1]*

Both of these work through the same lemma. Call a triple of models M_0, M_1, and M_2 an *amalgamation problem* when $M_0 \prec M_1$ and $M_0 \prec M_2$ (and we want to amalgamate them). For $X \in \mathcal{P}_\kappa(M_1 \cup M_2)$, fix $M^X_\ell \prec M_\ell$ such that $M^X_0 \prec M^X_1, M^X_2$. This is a 'small approximation' of the problem. A solution of an amalgamation problem is an amalgam.

Lemma 4.10. *If κ is strongly compact and every small approximation has a solution, then so does the original problem.*

Proof: For each X, witness the existence of a solution by $f_\ell^X : M_\ell^X \to N^X$ such that $f_1^X \restriction M_0^X = f_2^X \restriction M_0^X$. Let U be a fine, κ-complete ultrafilter on $\mathcal{P}_\kappa(M_1 \cup M_2)$. Now take the ultraproduct

$$\begin{tikzcd}
 & \prod M_1^X/U \arrow[r, "\prod f^X/U"] & \prod N^X/U \\
M_1 \arrow[ur, "\jmath"] & \prod M_0^X/U \arrow[u] \arrow[r] & \prod M_2^X/U \arrow[u, "\prod g^X/U"'] \\
M_0 \arrow[u] \arrow[ur, "\jmath"] \arrow[r] & M_2 \arrow[ur, "\jmath"] &
\end{tikzcd}$$

where $\jmath$ is the map that takes m to $[X \mapsto m]_U$. This is well-defined for U-many X exactly by fineness and an argument similar to Theorem 4.4 shows that it is a strong embedding. Then we have the desired amalgam! †

Proof of 4.9: (2) follows immediately.

For (1), call $M \in \mathcal{K}$ *κ-existentially closed*[8] or κ-e.c. iff for every $M_0 \prec M$ and $N_0 \succ M_0$ with $\|N_0\| < \kappa$, if there is $N \succ M$ and $f : N_0 \to_{M_0} N$, then there is $g : N_0 \to_{M_0} M$. Then Lemma 4.10 implies that κ-e.c. models are amalgamation bases. We prove that $\mathcal{K}_{\geq\kappa}$ has AP by showing every model is an amalgamation base in three steps:

- **In $\mathcal{K}_\mu$:** First, an exercise!

  Exercise 4.11. *Show that there is a κ-e.c. model of size $\lambda = \lambda^{<\kappa}$ for any λ.*

  Then this plus categoricity implies every model in $\mathcal{K}_\mu$ is an amalgamation base.
- **In $\mathcal{K}_{>\mu}$:** Suppose that $M \in \mathcal{K}_{>\mu}$ were not κ-e.c. Then there is $M_0 \prec M$ and $N_0 \succ M_0$ that witnesses this. Any submodel of M containing M_0 is then also not κ-e.c. with the same witness. In particular, there is a submodel of M of size μ containing M_0, contradicting the first step.
- **In $\mathcal{K}_{[\kappa,\mu)}$:** Given an amalgamation problem M_0, M_1, M_2, find a κ-complete ultrafilter (or even coherent ultrafilter/extender) U such that $\|\prod M_0/U\| \geq \mu$. Then amalgamate $\prod M_1/U$ and $\prod M_2/U$ over $\prod M_0/U$.

†

4.1.2. *Universal Classes.* We now turn to proving tameness in a particularly nice class of AECs, universal classes. These include many nice algebraic examples like locally finite groups (recall Example 2.7.(2)). We only touch on tameness here, but these have seen a lot of attention, especially Vasey [Vas17b, Vas17c] that proves Shelah's Categoricity Conjecture for them (see also Shelah [She09b, Chapter V]).

Definition 4.12. *An AEC $\mathcal{K}$ is a* universal class *iff*

- *$\prec_\mathcal{K} = \subset$; and*
- *if $A \subset M$ is closed under the functions of M, then $A \in \mathcal{K}$ (and $A \prec_\mathcal{K} M$).*

Given $A \subset M \in \mathcal{K}$, write $\langle A \rangle^M$ for the τ-structure that is the closure of A under the functions of M.

Although it's unlikely Tarski every considered AECs [citation needed], a result of his essentially gives a syntactic characterization of universal classes.

[8]In [Bon14], I foolishly called these κ-universally closed models, even noting that they are 'a generalization of existential closure' [Bon14, p. 1113].

Theorem 4.13 (Tarski, [Tar54])**.** *Any universal class $\mathcal{K}$ is axiomatized by $T = \{\phi_n \mid n < \omega\}$ from $\mathbb{L}_{(I(\mathcal{K},|\tau|)+|\tau|+\omega)^+,\omega}$, where ϕ_n is of the form*

$$\forall x_1, \dots x_n \bigvee_i \bigwedge_j \phi_n^{i,j}(x_1, \dots, x_n)$$

where $\phi_n^{i,j}$ is basic.

Proof: As in the case of locally finite groups, '[t]he proof is easier to see than to write down' [MS76, p. 171]. The idea is that ϕ_n says the closure of each n-tuple under functions is isomorphic to some structure in $\mathcal{K}_{|\tau|}$. Something like

$$\phi_n :\equiv \forall \mathbf{x} \bigvee_{\substack{\mathbf{m} \in M_0 \in \mathcal{K}_{|\tau|} \\ \langle \mathbf{m} \rangle = M_0}} \bigwedge \langle \mathbf{x} \rangle \cong \langle \mathbf{m} \rangle$$

†

The key to universal classes is that the generation of substructures by closure under functions is canonical. In turn, given any assignment of tuples $\mathbf{a} \mapsto \mathbf{b}$, there is only one way to extend this map to a map between their closure (or only one way to do this that has any chance of preserving τ-structure). We exploit this in the following theorem, and later use an example of Baldwin and Shelah [BS08] to show that this canonicity is necessary.

Proposition 4.14 (B.)**.** *Any universal class is $< \omega$-type short over the empty set (and therefore, fully $< \omega$-tame and -type short and compact!).*

Proof: Let $\langle a_i^\ell \mid i < \lambda \rangle \in N_\ell$ for $\ell = 1, 2$ such that for any $X \in \mathcal{P}_\omega \lambda$, we have

$$p^X = \text{gtp}(\langle a_i^1 \mid i \in X \rangle / \emptyset; N_1) = \text{gtp}(\langle a_i^2 \mid i \in X \rangle / \emptyset; N_2) = q^X$$

Write $\mathbf{a}_X^\ell$ for $\langle a_i^\ell \mid i \in I \rangle$. This means there is $f_X : \langle \mathbf{a}_X^1 \rangle^{N_1} \cong \langle \mathbf{a}_X^2 \rangle^{N_2}$ and that this map is determined by the fact it extends a_i^1 to a_i^2. In particular, whenever $X \subset Y$, the following diagram commutes

$$\begin{tikzcd}
\langle \mathbf{a}_Y^1 \rangle^{N_1} \arrow[r, "f_Y", "\cong"'] & \langle \mathbf{a}_Y^2 \rangle^{N_2} \\
\langle \mathbf{a}_X^1 \rangle^{N_1} \arrow[r, "f_X", "\cong"'] \arrow[u] & \langle \mathbf{a}_X^2 \rangle^{N_3} \arrow[u]
\end{tikzcd}$$

Thus, $\{f_X \mid X \in \mathcal{P}_\omega \lambda\}$ is a directed system of maps between the directed systems $\{\langle \mathbf{a}_X^1 \rangle^{N_1} \mid X \in \mathcal{P}_\omega \lambda\}$ and $\{\langle \mathbf{a}_X^2 \rangle^{N_2} \mid X \in \mathcal{P}_\omega \lambda\}$. Thus, taking the colimit (=directed unions), we get

$$f_\lambda := \bigcup_{X \in \mathcal{P}_\omega \lambda} f_X : \langle \mathbf{a}_\lambda^1 \rangle^{N_1} \cong \langle \mathbf{a}_\lambda^2 \rangle^{N_2}$$

Thus, we have the following diagram to witness type equality.

$$\begin{tikzcd}
 & N_1 & & N_2 & \\
N_1 \arrow[ur] & \langle \mathbf{a}_\lambda^1 \rangle^{N_1} \arrow[u] \arrow[rr, "f_\lambda", "\cong"'] & & \langle \mathbf{a}_\lambda^2 \rangle^{N_2} \arrow[u] & N_2 \arrow[ul] \\
 & & M \arrow[ull] \arrow[ul] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

†

The canonicity is crucial! The ideas of 'sub-X generated by' and 'class of X is closed under intersection' are closely linked, but the canonicity endowed by the first is crucial to the above argument. We make this precise with an example of Baldwin and Shelah [BS08]. We're lighter on references than other areas, but most of these ideas can be found in [BS08, **?**]. The main takeaway is the following:

Theorem 4.15 (Baldwin-Shelah, [BS08]). *There is an AEC $\mathcal{K}^{ses}$ with the following properties:*

(1) *$\mathcal{K}^{ses}$ is not $(\aleph_0, \aleph_1)$-tame.*
(2) *If $V = L$, then $\mathcal{K}^{ses}$ is not $(< \kappa, \kappa)$-tame for any regular, uncountable, not weakly compact κ.*

We review some basic facts from algebra.

Remark 4.16. *Throughout the rest of this section, 'group' means 'abelian group.' In particular, 'free group' means 'free abelian group.'*

A *short exact sequence of groups* is a collection of groups and group homomorphisms

$$0 \longrightarrow G_0 \xrightarrow{f} G_1 \xrightarrow{g} G_2 \longrightarrow 0$$

where f is injective, g is surjective, and $\ker g = \text{im } f$ In a sense, it's a way to express that $G_2 \cong G_1/f(G_0)$. The easiest way to build one is to take your favorite G_0 and G_2 and set $G_1 = G_0 \oplus G_2$. If your favorite G_0 is $\mathbb{Z}$ and this easy way is the *only* way to build a short exact sequence, then your G_2 is Whitehead.

Definition 4.17. *G is a* Whitehead group *(or W-group) iff for short exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} H \xrightarrow{g} G \longrightarrow 0$$

splits; that is, there is $u : G \to H$ such that $g \circ u = id$.

The classic question about Whitehead groups is of course Whitehead's Problem ([**?**], but never cite Wikipedia): is every Whitehead group free?

Recall that

- a group is free iff it has a set of generators with no nontrivial relations between them
- a group is κ-free iff every subgroup of size $< \kappa$ is free
- a group G is almost free iff it is not free but is $|G|$-free.

When first learning about these concepts, the idea of an $\aleph_1$-free, not free group seemed too strange. Couldn't, young Will reasoned, you take whatever relation witnessed non-freeness, form the subgroup generated by the involved elements, and get a countable non-free subgroup? The problem is that the choice of generators is not fixed. So the countable subgroup might have some different set of elements who generate it in such a way that the bad relation becomes trivial.

Standard results show that free implies Whitehead implies $\aleph_1$-free. Shelah's surprising answer to the Whitehead problem is that it is independent.

Fact 4.18 ([**?**, **?**]).

(1) *If $\diamondsuit_{\aleph_1}$, then every Whitehead group of size $\aleph_1$ is free.*
(2) *If MA holds and $2^{\aleph_0} > \aleph_1$, then there is a non-free Whitehead group of size $\aleph_1$.*

For Baldwin and Shelah's construction, we are more interested in almost free, non-Whitehead groups.

Fact 4.19.

(1) *There is an almost free, non-Whitehead group of size $\aleph_1$.*

(2) *In L, there is an almost free, non-Whitehead group of size κ for every regular, uncountable κ that is not weakly compact.*
(3) *Almost free implies free at singular cardinals, at weakly compact cardinals, and above strongly compact cardinals.*

(1) is due to Shelah [**?**, p. 228]. (2) is due to an analysis of the proof of (1) (see [Bon14, Section 8]): nonreflecting stationary sets are used to build almost free groups, and weak diamond [**?**] on every stationary set is used to show that every Whitehead group is free. The first exist at precisely uncountable, regular, not weakly compact cardinals in L, and the second holds in L. The first item of (3) is Shelah's Singular Compactness Theorem [**?**] and the other two items are easy applications of compactness.

Now we begin Baldwin and Shelah's construction.

Define $\mathcal{K}^{ses}$ to consist of models M that code families of short exact sequences that start with $\mathbb{Z}$ and end in a common group $G = G^M$:

$$\begin{tikzcd}
 & & H_i \arrow[dr, "g_i"] & & \\
0 \arrow[r] & \mathbb{Z} \arrow[ur, "f_i"] \arrow[r, "f_j"] \arrow[dr, "f_k"'] & H_j \arrow[r, "g_j"] & G \arrow[r] & 0 \\
 & & H_k \arrow[ur, "g_k"'] & &
\end{tikzcd}$$

This is coded by having a set of indices I and surjection $\pi : H \to I$ so that H_i is shorthand for $\pi^{-1}\{i\}$. There is a binary $+$ that is a group operation on $\mathbb{Z}$, G, and each H_i individually. Also, the f_i's are formally coded as binary functions $f : I \times \mathbb{Z} \to H$ (and the g_i's are a single $g : H \to G$), but we won't write this.

Having defined the models, for $M, N \in \mathcal{K}^{ses}$, we write $M \prec_{ses} N$ iff $M \subset N$, they have the same copy of $\mathbb{Z}$, and G^M is a pure subgroup of G^N; this means that if $g \in G^M$ has an nth root in G^N, it also has one in G^M.

The following are nice to know, but not crucial.

Fact 4.20 ([BS08, Lemmas 2.5 and 2.7])**.** *$\mathcal{K}^{ses}$ has amalgamation and is closed under intersections*[9]*.*

A class being closed under intersections has a long history in elementary classes, but Baldwin and Shelah were the first to isolate it for AECs.

Definition 4.21 ([BS08, Definition 1.2])**.** *$\mathcal{K}$ is* closed under intersections *iff for every $A \subset M \in \mathcal{K}$,*

$$\bigcap\{N \in \mathcal{K} \mid A \subset N \prec M\} \prec M$$

Write $cl_M(A)$ for the first set (or rather the substructure of M with that universe).

Closure under intersection gives a very nice condition to check type equality [BS08,], which specializes to the following lemma in $\mathcal{K}^{ses}$.

Lemma 4.22 ([BS08, Lemma 2.6])**.** *Let $M \prec N_1, N_2$ with*
- *$G^M = G^{N_\ell}$; and*
- *$i_\ell \in I^{N_\ell} - I^M$.*

[9] This is not formally true, but only for trivial reasons. For instance, the $\emptyset$ has no minimal structure containing it because you need to make a choice of a middle sequence to include. This can be remedied by adding a constant in I or allowing empty I.

Then $gtp(i_1/M;N_1) = gtp(i_2/M;N_2)$ *iff there is some* $h : H^{N_1}_{i_1} \cong H^{N_2}_{i_2}$ *that commutes with the short exact sequences; that is,*

(1) *for* $n \in \mathbb{Z}$, $h\left(f^{N_1}_{i_1}(n)\right) = f^{N_2}_{i_2}(n)$*; and*

(2) *for* $x \in H^{N_1}_{i_1}$, $g^{N_1}_{i_1}(x) = g^{N_2}_{i_2}(h(x))$.

In particular, this implies that type equality means that the short exact sequence associate with i_1 splits iff the short exact sequence associated with i_2 splits. Moreover, if both sequences split, then we have type equality.

Proof: Straightforward after observing that $\text{cl}_{N_\ell}(Mi_\ell)$ only adds $H^{N_\ell}_{i_\ell}$. †

Theorem 4.23 ([BS08, Theorem 2.8])**.** *Suppose that* G *is an almost free, non-Whitehead group of size* κ*. Then,* $\mathcal{K}$ *is not* $(<\kappa,\kappa)$*-tame.*

Proof: Let G be the desired group and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f_*} H_* \xrightarrow{g_*} G \longrightarrow 0$$

be a short exact sequence witnessing it is not Whitehead. Define $M_0 \prec_{ses} M_1, M_2$ as follows:

- M_0 is the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f_0} \mathbb{Z}\oplus G \xrightarrow{g_0} G \longrightarrow 0$$

  with index 0.

- M_1 adds another copy with index 1

$$\begin{tikzcd} & & \mathbb{Z}\oplus G \arrow[dr, "g_1"] & & \\ 0 \arrow[r] & \mathbb{Z} \arrow[ur, "f_1"] \arrow[r, "f_0"] & \mathbb{Z}\oplus G \arrow[r, "g_0"] & G \arrow[r] & 0 \end{tikzcd}$$

- M_2 extends the s. e. s. with H with index $*$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[dr, "f_*"] \arrow[r, "f_0"] & \mathbb{Z}\oplus G \arrow[r, "g_0"] & G \arrow[r] & 0 \\ & & H_* \arrow[ur, "g_*"] & & \end{tikzcd}$$

Then, $\text{gtp}(*/M;N_2) \neq \text{gtp}(1/M;N_1)$; otherwise, Lemma 4.22 would imply that the H_* s. e. s. splits iff the $\mathbb{Z}\oplus G$ one does, which is a contradiction. However, suppose that $M_0^- \prec M$ had size $<\kappa$. Then, $G^{M_0^-}$ is free and, in particular, Whitehead. Thus, when looking at $H_\ell^{cl_{N_\ell}(G^{M_0^-}\ell)} := g_\ell^{-1}(G^{M_0^-})$, these short exact sequences must split, even for $\ell = 2$. Then Lemma 4.22 implies

$$\text{gtp}\left(1/M_0^-; cl_{N_1}(G^{M_0^-}1)\right) = \text{gtp}\left(*/M_0^-; cl_{N_2}(G^{M_0^-}*)\right)$$

†

Proof of 4.15: Combine Fact 4.19 and Theorem 4.23. †

Some other theorems of note regarding failure of type locality:

Fact 4.24. (1) *If* $2^{\aleph_0} = \aleph_1$, $\diamondsuit_{\aleph_1}$, $\square_{\aleph_1}$, *and* $\diamondsuit_{S^{\aleph_2}_{cof\,\omega_1}}$, *then* $\mathcal{K}^{ses}$ *is either not* $(\aleph_1,\aleph_1)$*-compact or not* $(\aleph_2,\aleph_2)$*-compact. [BS08, Theorem 3.3]*[10]

[10] [BS08] writes $\square_{\aleph_2}$ in the hypothesis, but the definition is for what is normally called $\square_{\aleph_1}$.

(2)

4.1.3. *High Categoricity.* The final way to get some form of tameness is the following theorem of Shelah that was a key step in the proof of Theorem 3.6.(1). The results originally appeared in [She99][11] and were given a nice expositional account in [Bal09], which we draw on heavily (and often cite from). This exposition benefitted from work of Baldwin, Hyttinen, Shelah, and others.

Throughout, fix an AEC $\mathcal{K}$ and set $H_1 := \beth_{(2^{\text{LS}(\mathcal{K})})^+}$ to be its Hanf number. The following is [Bal09, Theorem 11.15] with some improvements due to Vasey (see the discussion after the proof).

Theorem 4.25. *If $\mathcal{K}$ is λ-categorical for $\lambda \geq H_1$, then there is $\chi < H_1$ such that $\mathcal{K}$ is weakly $(\chi, [H_1, \lambda))$-tame.*

This theorem relies on [Bal09, Lemma 11.14], which gives some measure of tameness to EM models. We give a slight improvement of this (further work in this direction will appear in [**?**]).

First, we define a property of blueprints called being presentation friendly. The exact statement is below, but essentially says that the blueprint Φ comes from the argument the proved Theorem 3.23 using Shelah's Presentation Theorem and Morley's Omitting Types Theorem. This is implicit in the proof in [Bal09].

Definition 4.26. $\Phi \in \Upsilon[\mathcal{K}]$ *is* presentation friendly *iff there are distinguished function symbols* $\{F_{n,\alpha} \mid n < \omega, \alpha < LS(\mathcal{K})\} \subset \tau(\Phi)$ *and a collection of quantifier-free types* Γ *in* $\tau \cup \{F_{n,\alpha} \mid n < \omega, \alpha < LS(\mathcal{K})\}$ *such that*

(1) *For any τ-structure M,*

$$M \in \mathcal{K} \iff \text{there is an expansion } M^* \text{ of } M \text{ that omits } \Gamma$$

(2) *For any τ-structures $M \subset N$,*

$$M \prec_{\mathcal{K}} N \iff \text{for every expansion } M^* \text{ that omits } \Gamma, \text{ there is an expansion } N^* \text{ that omits } \Gamma \text{ with } M^* \subset N^*$$

(3) *For any linear order I, $EM(I, \Phi)$ omits Γ.*

Proposition 4.27. *If $\Upsilon[\mathcal{K}] \neq \emptyset$, then there is $\Phi \in \Upsilon_{LS(\mathcal{K})}[\mathcal{K}]$ that is presentation friendly.*

Proof: By the proof of Theorem 3.23. †

Call a linear order $(I, <)$ *fieldable* iff it is the reduct of an ordered field.

Exercise 4.28. *If I is fieldable, then any two intervals of the form (a, b) for $a, b \in I \cup \{-\infty, \infty\}$ are isomorphic. (Hint: Consider lines and $\frac{-1}{x}$.)*

[Bal09, p. 87] claims that this holds of transitive linear ordeings, but I'm not sure. It holds by definition for non-infinity endpoints. It is not the case that any transitive linear order is fieldable (the irrationals are a counter-example, [**?**, Exercise 2.37.(2) and Corollary 8.16]).

Theorem 4.29. *Let $\mathcal{K}$ be an AEC and $\Phi \in \Upsilon_{LS(\mathcal{K})}[\mathcal{K}]$ be presentation friendly. Let I be a fieldable linear order of size μ, $I \subset J$, and set $M = EM_\tau(I, \Phi)$ and $N = EM_\tau(J, \Phi)$.*

Suppose that $\mathbf{a}, \mathbf{b} \in N$ have the property: for every $M_0 \subset M$ of size $< H_1$, $gtp(\mathbf{a}/M_0; N) = gtp(\mathbf{b}/M_0; N)$. Then $gtp(\mathbf{a}/M; N) = gtp(\mathbf{b}/M; N)$.

[11] The reader should beware that the versions of this paper on Shelah's archive and on arχiv have been heavily revised from the published version.

Comparing with [Bal09, Lemma 11.14], we remove the assumption that N is saturated and of amalgamation in general; that I has an increasing sequence of length μ; and that I is an initial segment of J (the removal of amalgamation is most significant).

Proof: For ease of the write-up, we prove this for atomic equivalence; proving this for its transitive closure adds more technical difficulty, but no new concepts. Although this seems like the same as proving it with amalgamation, we crucially don't make use of any automorphisms of saturated models.

Set $\tau = \tau(\mathcal{K})$. Write $\mathbf{a} = \sigma(\mathbf{s}, \mathbf{t})$ and $\mathbf{b} = \rho(\mathbf{s}, \mathbf{t})$ where $\mathbf{s} \in I$ and $\mathbf{t} \in J - I$. $\mathbf{s}$ divide I into intervals $I_1, \dots, I_n$. Since I is fieldable, these are all isomorphic, say $f_j : I_1 \cong I_j$ with f_1 being the identity.

For each $\chi < H_1$, fix some $A_\chi = \{c^\chi_\alpha \mid \alpha < \chi\}$ (just an enumeration, not increasing) and set $K^\chi = \cup_{j \leq n} f_j" A_\chi$. Then $|K^\chi| = \chi$, so by assumption,

$$\operatorname{gtp}\left(\mathbf{a}/EM_\tau(K^\chi\mathbf{s}, \Phi); EM_\tau(K^\chi\mathbf{st}, \Phi)\right) = \operatorname{gtp}\left(\mathbf{a}/EM_\tau(K^\chi\mathbf{s}, \Phi); N\right) = \operatorname{gtp}\left(\mathbf{b}/EM_\tau(K^\chi\mathbf{s}, \Phi); N\right) = \operatorname{gtp}\left(\mathbf{b}/EM_\tau(K\right.$$

Thus, we can find $M_\chi \in \mathcal{K}_\chi$ with $EM_\tau(K^\chi\mathbf{st}, \Phi) \prec M_\chi$ and $G^\chi : EM_\tau(K^\chi\mathbf{st}, \Phi) \to_{EM_\tau(K^\chi\mathbf{s}, \Phi)} M_\chi$ such that $\mathbf{a} = G^\chi(\mathbf{b})$.

$$\begin{tikzcd}
EM_\tau(K^\chi\mathbf{st}, \Phi) \arrow[r] & M_\chi \\
EM_\tau(K^\chi\mathbf{s}, \Phi) \arrow[u] \arrow[r] & EM_\tau(K^\chi\mathbf{st}, \Phi) \arrow[u, "G^\chi"']
\end{tikzcd}$$

Set τ_* to be the language consisting of

$$\left(P_0, P_1, P_2, (F)_1, (F)_2, f^1, \dots, f^n, c_{s_1}, \dots, c_{s_{n-1}}, d^1_{t_1}, \dots, d^1_{t_m}, d^2_{t_1}, \dots, d^2_{t_m}\right)_{F \in \tau(\Phi)}$$

where P_k is unary; for each $F \in \tau(\Phi)$, both $(F)_1$ and $(F)_2$ are functions/relations with the same arity; f^k are partial unary functions; and the rest are constants.

We think of τ_* as consisting of two copies of $\tau(\Phi)$. If σ' is a $\tau(\Phi)$-term, then write $(\sigma')_\ell$ for the τ_*-term that is built from the same composition of function symbols, but replacing F with $(F)_\ell$.

We want to expand M_χ to a τ_*-structure M^*_χ.

- $$P_\ell^{M^*_\chi} = \begin{cases} EM_\tau(K^\chi\mathbf{s}, \Phi) & \ell = 0 \\ EM_\tau(K^\chi\mathbf{st}, \Phi) & \ell = 1 \\ G^\chi{}'' EM_\tau(K^\chi\mathbf{st}, \Phi) & \ell = 2 \end{cases}$$
- $(f^k)^{M^*_\chi} = f_k \restriction A_\chi$
- $c_{s_i}^{M^*_\chi} = s_i = G^\chi(s_i)$
- $d^1_{t_j} = t_j$
- $d^2_{t_j} = G^\chi(t_j)$
- for the 1-version of $\tau(\Phi)$, expand M_χ to a $\tau(\Phi)$-structure M^+_χ that omits Γ such that

  $$EM(K^\chi\mathbf{s}, \Phi) \subset EM(K^\chi\mathbf{st}, \Phi) \subset M^+_\chi$$

  This is possible because Φ is presentation friendly. Then, for $F \in \tau(\Phi)$, set

  $$(F)_1^{M^*_\chi} = F^{M^+_\chi}$$

- for the 2-version of $\tau(\Phi)$, similarly expand M_χ to a $\tau(\Phi)$-structure M_χ^{++} that omits Γ such that
$$EM(K^\chi \mathbf{s}, \Phi) \subset G^{\chi\prime\prime} EM(K^\chi \mathbf{st}, \Phi) \subset M_\chi^{++}$$
Then, for $F \in \tau(\Phi)$, set
$$(F)_2^{M_\chi^*} = F^{M_\chi^{++}}$$

Note that, by our construction (really by the fact G^χ fixes $EM_\tau(K^\chi \mathbf{s}, \Phi)$),
$$(F)_1^{M_\chi^*} \restriction P_0^{M_\chi^*} = (F)_2^{M_\chi^*} \restriction P_0^{M_\chi^*}$$

We've built a τ_*-structure M_χ^* for each $\chi < H_1$ that witness the Galois type equality of $\mathbf{a}$ and $\mathbf{b}$ over certain small submodels. We really only need the part of M_χ^* that is in the hull of A_χ, but that's immaterial.

Now, apply Morley's Omitting Types Theorem 3.33 (actually the version of it in Remark 3.34) to get a blueprint $\Psi = \{q_n \mid n < \omega\}$ such that $\tau(\Psi) = \tau_*$ and for every $n < \omega$, there are cofinally many χ and $e_1^n < \cdots < e_n^n \in A_\chi$ such that
$$q_n = tp_{qf}\left(e_1^n, \ldots, e_n^n / \emptyset; M_\chi^*\right)$$

Set $N_* = EM(I_1, \Psi)$. Since each M_*^χ omits Γ, so does N_*. Thus, $N_* \restriction \tau \in \mathcal{K}$. Moreover, the same is true of each $P_\ell^{N_*}$, so $P_0^{N_*} \restriction \tau \prec P_k^{N_*} \restriction \tau \prec N_* \restriction \tau$ for $k = 1, 2$. Now we want to build strong embeddings to make the following diagram commute

$$\begin{tikzcd}
 & P_1^{N_*} \restriction \tau \arrow[r] & N_* \restriction \tau = EM_\tau(I_1, \Psi) \\
EM_\tau(I\mathbf{t}, \Phi) \arrow[ur, "g_1"] & P_0^{N_*} \restriction \tau \arrow[u] \arrow[r] & P_2^{N_*} \restriction \tau \arrow[u] \\
EM_\tau(I, \Phi) \arrow[u] \arrow[ur, "g_0"] \arrow[r] & EM_\tau(I\mathbf{t}, \Phi) \arrow[ur, "g_2"] &
\end{tikzcd}$$

(1) if $i \in I_1$, then $g_\ell(i) = i$;
(2) if $i \in I_k$, then $g_\ell(i) = (f^k)^{N_*} \circ f_k(i)$;
(3) $g_\ell(s_i) = c_{s_i}^{N_*}$;
(4) $g_1(t_j) = (d_{t_j}^1)^{N_*}$;
(5) $g_2(t_j) = (d_{t_j}^2)^{N_*}$; and
(6) if σ' is a $\tau(\Phi)$-term and $i_1 < \cdots < i_q \in I\mathbf{t}$, then
$$g_\ell\left(\sigma'(i_1, \ldots, i_q)\right) = (\sigma')_\ell\left(g_\ell(i_1), \ldots, g_\ell(i_q)\right)$$

Finally, note that (recalling our representation of $\mathbf{a}$ and $\mathbf{b}$), for each χ,
$$\sigma^{EM(K^\chi \mathbf{st}, \Phi)}(s_1, \ldots, s_n; t_1, \ldots, t_m) = G^\chi(\rho^{EM(K^\chi \mathbf{st}, \Phi)}(s_1, \ldots, s_n; t_1, \ldots, t_m)$$

Thus,
$$`(\sigma)_1\left(c_{s_1}, \ldots, c_{s_n}; d_{t_1}^1, \ldots, d_{t_m}^1\right) = (\rho)_2\left(c_{s_1}, \ldots, c_{s_n}; d_{t_1}^2, \ldots, d_{t_m}^2\right)' \in q_k$$

Moreover, $g_1(\mathbf{a})$ is sent to the interpretation of the left hand side and $g_2(\mathbf{b})$ is sent to the interpretation of the right hand side. Thus, $g_1(\mathbf{a}) = g_2(\mathbf{b})$ and
$$\mathrm{gtp}\left(\mathbf{a}/EM_\tau(I, \Phi); EM_\tau(I\mathbf{t}, \Phi)\right) = \mathrm{gtp}\left(\mathbf{b}/EM_\tau(I, \Phi); EM_\tau(I\mathbf{t}, \Phi)\right)$$

as desired. †

By finding blueprints that output fieldable linear orders containing I, we can show the following.

Corollary 4.30. *There is a blueprint...*

Now that we know that certain EM models are 'tameness bases,' we turn to the problem of recognizing them. This is where saturation comes in. We will show that every EM model is saturated (under a suitable hypothesis). Then, the uniqueness of saturated models implies that every saturated model is one of our tameness bases, thus implying weak tameness. We have seen how to build Galois saturated models from Galois stability (Proposition 3.16.(2)). The following is a straightforward application of this.

Exercise 4.31. *If $\mathcal{K}$ is λ-Galois stable, then there is a cf λ-Galois saturated model of size λ.*

This is the source of the cf λ in other versions of this result (e. g., [Bal09, Theorem 11.15]). When λ is regular (in particular, a successor), this gives a Galois saturated model. Baldwin and Shelah both asked if this could be removed by showing that the categoricity model is always saturated even if λ is singular. Vasey has answered this in the affirmative.

Fact 4.32 ([Vas17a, Corollary 4.11.(3)])**.** *Let $\lambda > LS(\mathcal{K})$ and suppose that $\mathcal{K}_{<\lambda}$ has amalgamation and no maximal models, and that $\mathcal{K}$ has arbitrarily large models. If $\mathcal{K}$ is categorical in λ, then the model of size λ is Galois-saturated.*

This allows us to show that categoricity implies that many EM models are saturated. We are going to use this result heavily, so must be careful to not use it in a result that shows it. In particular, we can't use it to prove Theorem 5.50.

Lemma 4.33. *If $\mathcal{K}$ is λ-categorical, has amalgamation, $\Phi \in \Upsilon_{LS(\mathcal{K})}[\mathcal{K}]$, $\theta < \lambda$, and J is a linear order containing an increasing sequence of length θ^+, then $EM_\tau(J, \Phi)$ is θ^+-Galois saturated.*

Proof: Let $M \prec EM_\tau(J, \Phi)$ of size $\leq \theta$ and $p \in \text{gS}(M)$. Set $\langle a_i \mid i < \theta^+ \rangle \subset J$ be increasing and set $a_* = \sup a_i$. We can decompose J into a sum $J_0 + J_1$ with a_* the first element of J_1. Then J is naturally a suborder of $J^* := J_0 + \lambda + J_1$. $|J^*| = \lambda$, so by Fact 4.32 $EM_\tau(J^*, \Phi)$ is Galois saturated. Thus, there is $a \in EM_\tau(J^*, \Phi)$ that realizes p and we can write $p = \sigma(\mathbf{j}_0, \mathbf{i}, \mathbf{j}_1)$ for σ a $\tau(\Phi)$-term, $\mathbf{j}_\ell \in J_\ell$, and $\mathbf{i} \in \lambda$, all written in increasing order.

Recall that Contents(M) is the/a minimal $J_M \subset J$ such that $M \prec EM_\tau(J_M, \Phi)$. Set

$$i_0 := \min\left\{i < \theta^+ \mid \forall j \in \text{Contents}(M) \cup \{\mathbf{j}_0\}(j > a_i \to j \geq a_*)\right\}$$

This set is nonempty by a cofinality argument. In fact, there are still μ^+ many elements of the sequence above i_0. Thus, there are $\mathbf{i'} < \theta^+$ such that

$$tp_{qf}\left(\text{Contents}(M)\mathbf{j}_0\mathbf{i}\mathbf{j}_1/\emptyset; J^*\right) = tp_{qf}\left(\text{Contents}(M)\mathbf{j}_0\mathbf{a}_{\mathbf{i'}}\mathbf{j}_1/\emptyset; J\right)$$

By the EM construction, this tells us

$$\text{gtp}\left(\sigma(\mathbf{j}_0, \mathbf{i}, \mathbf{j}_i)/M; EM_\tau(J^*, \Phi)\right) = \text{gtp}\left(\sigma(\mathbf{j}_0, \mathbf{a}_{\mathbf{i'}}, \mathbf{j}_1)/M; EM_\tau(J, \Phi)\right)$$

In particular, $EM_\tau(J, \Phi)$ realizes p.

Since M was arbitrary, $EM_\tau(J, \Phi)$ is θ^+-Galois saturated. †

Corollary 4.34. *If $\mathcal{K}$ is λ-categorical, has amalgamation, $\Phi \in \Upsilon_{LS(\mathcal{K})}[\mathcal{K}]$, and J is a linear order containing $|J|$ as a suborder with $|J| \leq \lambda$, then $EM_\tau(J, \Phi)$ is Galois saturated.*

Now we prove our goal, Theorem 4.25.

Proof of 4.25: First, we show that $\mathcal{K}$ is $(< H_1, [H_1, \lambda))$-weakly tame. Let $\mu \in [H_1, \lambda)$, $M \in \mathcal{K}_\mu$-Galois saturated, and $p \neq q \in \text{gS}(M)$. Set J_μ to be a fieldable linear order of size μ that contains μ as a suborder. Then, by Corollary 4.34, $EM_\tau(J_\mu, \Phi)$ is Galois saturated. WLOG, $M = EM_\tau(J_\mu, \Phi)$. Then $EM_\tau(J_\mu + \lambda, \Phi)$ is Galois saturated and larger, so it contains realizations of p and q. By Theorem 4.29, there must be some $< H_1$-sized model witnessing their difference.

Now we improve the bound. Let $M \in \mathcal{K}_{H_1}$ be Galois saturated. WLOG $M = EM_\tau(I, \Phi)$ for I fieldable and every type over it is realized in $EM_\tau(I + \lambda, \Phi)$ by the above paragraph. By Theorem 4.29, for every $p \neq q \in \mathrm{gS}(M)$, there is $\chi_{p,q} < H_1$ such that $p \upharpoonright M_0 \neq q \upharpoonright M_0$ for some M_0 of size $\chi_{p,q}$. Then there are terms σ and ρ and $\mathbf{s} \in I$ and $\mathbf{t} \in \lambda$ such that $\sigma(\mathbf{s}, \mathbf{t})$ realizes p and $\rho(\mathbf{s}, \mathbf{t})$ realizes q. Note that if $p' \neq q' \in \mathrm{gS}(M)$ are realized by the same pair of terms (although with different inputs), then the structure of the orders implies that $p' \upharpoonright M_0' \neq q' \upharpoonright M_0'$ for some model M_0' of the same size as M_0. This uses the transitivity of the linear order I and the fact that I is an initial segment of λ. So $\chi_{p,q}$ is an invariant of the terms used (so write it $\chi_{\sigma,\rho}$). Set $\chi = \sup_{\sigma,\rho \in \tau(\Phi)} \chi_{\sigma,\rho}$. Then by cofinality, $\chi < H_1$ and, by construction, $\mathcal{K}$ is weakly (χ, H_1)-tame.

The fact that this χ works for all $\mu \in [H_1, \lambda)$ follows from the following exercise.

Exercise 4.35. *Suppose $\mathcal{K}$ has no maximal models. If $\mathcal{K}$ is (κ, μ)-tame and (μ, λ)-tame, then it is (κ, λ)-tame.*

†

The observation that you can fix a single cardinal below H_1 is due to Vasey and answers [Bal09, Question 11.16] (see [Vas17a, Corollary 5.7.(5)]).

5. Categoricity transfer

Now that we have seen where tameness occurs, we return to our goal of proving categoricity transfer in various contexts from tameness (Theorem 3.6). We continue to follow the presentation in [Bal09].

5.1. **A weak independence notion.** .

Suppose that $f : M \to N$ is a strong embedding. Then there is a canonical map from $\mathrm{gS}(M)$ to $\mathrm{gS}(f(M))$ which we also write as f that is defined as follows: suppose that $p = \mathrm{gtp}(a/M; N_1) \in \mathrm{gS}(M)$. Then we can find an extension N_2 of $f(M)$ and an isomorphism $f^* : N_1 \cong N_2$ that extends f. Then set

$$f(p) := \mathrm{gtp}\left(f^*(a)/f(M); N_2\right)$$

Exercise 5.1. *The above map is well-defined.*

The following notion of Galois splitting (or simply splitting) was introduced by Shelah [She99, Definition 3.2] (I think) and greatly expanded upon by VanDieren [Van06] and later many others. One such extension is Vasey's use to derive a good frame from categoricity in the presence of tameness (see Theorem **??**).

Definition 5.2. *Let $M \in \mathcal{K}_{\leq\mu}$, $N \succ M$, and $p \in gS(N)$. We say p μ-*Galois splits *over M iff there are $N_1, N_2 \in \mathcal{K}_\mu$ with $M \prec N_\ell \prec N$ and $h : N_1 \cong_M N_2$ such that $p \upharpoonright N_2 \neq h(p \upharpoonright N_1)$.*

Of course, Galois nonsplitting is the negation of the above. Some basic properties of this are obvious.

Exercise 5.3. *Nonsplitting satisfies the following properties:*

1. *Invariance: If $p \in gS(N)$ does not μ-Galois split over M and $f : N \cong N'$, then $f(p)$ does not μ-Galois split over $f(M)$.*
2. *Monotonicity: If $p \in gS(N)$ does not μ-Galois split over M and $M \prec M' \prec N' \prec N$ with $\|M\| \leq \mu$, then $p \upharpoonright N'$ does not μ-Galois split over M'.*

We want to explore some more complicated properties. There are three initial nice properties that we will discuss: existence of nonsplitting bases, uniqueness of nonsplitting extensions, and extension of nonsplitting types (VanDieren [**?**, Definition 3] recently introduced a symmetry property). Note that wether or not it is transitive seems unknown [Bal09, Errata to Exercise 12.9].

Before beginning, we should mention some motivation. Recall that, in first-order, a (syntactic) type $p \in S(A)$ does not split over $A_0 \subset A$ iff for all $\mathbf{b}_1, \mathbf{b}_2 \in A$, if $tp(\mathbf{b}_1/A_0) = tp(\mathbf{b}_2/A_0)$, then for all formulas $\phi(\mathbf{x}, \mathbf{y})$, $\phi(\mathbf{x}, \mathbf{b}_1) \in p$ iff $\phi(\mathbf{x}, \mathbf{b}_2) \in p$. This says that the way that the type interacts with some $\mathbf{b} \in A$ is completely determined by $tp(\mathbf{b}/A_0)$. Given some mild saturation of A, this allows us to build canonical (nonsplitting) extensions p^* of p to any $B \supset A$: let $\mathbf{b} \in B$. By mild saturation, find $\mathbf{c} \in A$ such that $tp(\mathbf{b}/A_0) = tp(\mathbf{c}/A_0)$. Then we set $p^* \upharpoonright \mathbf{b} = \{\phi(\mathbf{x}, \mathbf{b}) \mid \phi(\mathbf{x}, \mathbf{c}) \in p\}$. This completely determines p^*.

Now Galois nonsplitting looks a little difference, but the main difference is cosmetic (and that types are no longer syntactic objects determined finitarily). First note that, for models, $\text{gtp}(N_1/M; N) = \text{gtp}(N_2/M; N)$ iff $N_1 \cong_M N_2$ (this doesn't even need amalgamation). Then suppose that $p = tp(\mathbf{a}/A) \in S(A)$ and $A_0 \subset A$ and $\mathbf{b}_1, \mathbf{b}_2 \in A$ with the same type over A_0. Then $\text{id}_{A_0} \cup \{(\mathbf{b}_1, \mathbf{b}_2)\}$ is a partial elementary map. Moreover, the condition "$\phi(\mathbf{x}, \mathbf{b}_1) \in p$ iff $\phi(\mathbf{x}, \mathbf{b}_2) \in p$" is equivalent to the condition that

$$h(p \upharpoonright \mathbf{b}_1) = p \upharpoonright \mathbf{b}_2$$

We might ask how

[INTERACTION WITH TAMENESS]

Existence: First, we show that the existence of long splitting chains contradicts Galois stability. We work inside a monster model for ease (especially the extension of maps to automorphisms), but only really need amalgamation.

Lemma 5.4. *Assume $\mathcal{K}$ has a monster model and let $\mu \geq LS(\mathcal{K})$. If there is $M = \cup_{i<\mu} M_i$ for a continuous, increasing $\langle M_i \mid i < \mu \rangle$ from $\mathcal{K}_\mu$ and $p \in gS(M)$ such that $p \upharpoonright M_{i+1}$ μ-Galois splits over M_i for each $i < \mu$, then there is $M_* \in \mathcal{K}_{2^{<\mu}}$ such that $|gS(M_*)| \geq 2^\mu$.*

Proof: From the assumptions, there are $M_i \prec M_i^1, M_i^2 \prec M_{i+1}$ and $h_i : M_i^1 \cong_{M_i} M_i^2$ such that

$$h_i(p \upharpoonright M_i^1) \neq p \upharpoonright M_i^2$$

We are going to build a continuous increasing *tree* of models $\langle N_\eta \in \mathcal{K}_{\leq \kappa} \mid \eta \in {}^{<\mu}2 \rangle$, functions $f_\eta : M_{\ell(\eta)} \cong N_\eta$, and types $p_\eta \in \text{gS}(N_\eta)$ such that

(1) $p_\eta = f_\eta(p \upharpoonright M_{\ell(\eta)})$;
(2) $f_\eta(M_i^2) \prec N_{\eta \frown \langle \ell \rangle}$; and
(3)
$$p_{\eta \frown \langle 0 \rangle} \upharpoonright f_\eta(M_i^2) \neq p_{\eta \frown \langle 1 \rangle} \upharpoonright f_\eta(M_i^2)$$

Construction: $\eta = \emptyset$: $f_\eta = \text{id}$, $N_\eta = M_0$, $p_\eta = p \upharpoonright M_0$.

$\ell(\eta)$ is limit: Take unions.

$\eta = \nu^\frown\langle\ell\rangle$: Write α for $\ell(\nu)$. Now extend f_ν for $\hat{f}_\nu \in \mathrm{Aut}\,\mathfrak{C}$ and h_α to $\hat{h}_\alpha \in \mathrm{Aut}_{M_\alpha}\mathfrak{C}$. This gives us the following set-up:

$$\begin{tikzcd}
 & M_{\alpha+1} & & & \hat{f}_\nu(M_{\alpha+1}) & \\
M_\alpha^1 \arrow[ur] \arrow[rr, "h_\alpha", "\cong"'] & & M_\alpha^2 \arrow[ul] \arrow[r, Rightarrow, "\hat{f}_\nu"] & \hat{f}_\nu(M_\alpha^1) \arrow[ur] \arrow[rr, "\hat{f}_\nu \circ h_\alpha \circ \hat{f}_\nu^{-1}", "\cong"'] & & \hat{f}_\nu(M_\alpha^2) \arrow[ul] \\
 & M_\alpha \arrow[ul] \arrow[ur] & & & N_\nu \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Then set

$$\begin{aligned} h_{\nu^\frown\langle 0\rangle} &= \hat{f}_\nu \restriction M_{\alpha+1} \\ h_{\nu^\frown\langle 1\rangle} &= \hat{f}_\nu \circ \hat{h}_\alpha \restriction M_{\alpha+1} \end{aligned}$$

By the splitting, $h_\alpha(p \restriction M_\alpha^1) \neq p \restriction M_\alpha^2$. Applying f_ν, this gives

$$\begin{aligned} \hat{f}_\nu \circ \hat{h}_\alpha(p \restriction M_\alpha^1) &\neq \hat{f}_\nu(p \restriction M_\alpha^2) \\ p_{\nu^\frown\langle 1\rangle} \restriction \hat{f}_\nu(M_\alpha^2) &\neq p_{\nu^\frown\langle 0\rangle} \restriction \hat{f}_\nu(M_\alpha^2) \end{aligned}$$

as desired.

This is enough: Find $M_* \prec \mathfrak{C}$ of size κ such that $N_\eta \prec M_*$ for all $\eta \in {}^{<\mu}2$. For $\eta \in {}^{\mu}2$, set

$$p_\eta := \left(\bigcup_{\alpha<\mu} h_{\eta\restriction\alpha}\right)(p)$$

By part (3) of the construction, each of these types are distinct, so there $\geq 2^\mu > \kappa$ many of them.†

Theorem 5.5 (Existence [She99, Claim 3.3])**.** *Suppose $\mathcal{K}$ has a monster model. If $M \in \mathcal{K}_{\geq\mu}$ and $\mathcal{K}$ is μ-Galois stable, then for every $p \in gS(M)$, there is $N_p \in \mathcal{K}_\mu$ such that $N_p \prec M$ and p does not μ-Galois split over N_p.*

Proof: Easy if $\|M\| = \mu$. If $\|M\| > \mu$, then build a chain as in Lemma 5.4: pick $M_0 \prec M$ to be arbitrary of size μ. Given M_i, p μ-Galois splits over it, so let $M_{i+1} \prec M$ of size μ contain the witnesses. †

Uniqueness: To get uniqueness of nonsplitting extensions to models of larger size, we need to make an important use of tameness. First, we define universal models. This will correspond to the mild saturation assumption mentioned when discussing first-order splitting.

Definition 5.6. *For $M \prec N$, we say N* is κ-universal over M *iff $\|M\| \leq \kappa$ and for all $M^* \succ M$ from $\mathcal{K}_{\leq\kappa}$, there is $f : M^* \to_M N$.*

A common trick we will see is that if N is κ-universal over M and also of size κ, then any further extension of N of size κ can be embedded into N over M.

Theorem 5.7 (Uniqueness, [Van06, Theorem I.4.12])**.** *Let $\mathcal{K}$ be (κ, λ)-tame, $\kappa \leq \mu \leq \lambda$, and $M_0 \prec M_1 \prec N$ with $p \in gS(M_1)$ such that*

(1) *$N \in \mathcal{K}_\lambda$;*
(2) *M_1 is μ-universal over M_0; and*
(3) *p does not μ-Galois split over M_0.*

Then, there is at most one extension of p to N that does not μ-Galois split over M_0.

Note that, for the case $\kappa = \mu = \lambda$, every $\mathcal{K}$ is (κ, κ)-tame. Really, as we will see in the proof, the idea is that M_2 needs to be a '(κ, λ)-tameness base,' so if M_2 is Galois saturated, then the corresponding weak tameness is enough.

Proof: Suppose that $q \neq r \in \mathrm{gS}(N)$ do not μ-Galois split over M_0 and both extend p. By tameness, find $N_0 \prec N$ of size κ such that $q \upharpoonright N_0 \neq r \upharpoonright N_0$. Then find $N_1 \prec N$ of size μ extending M_0 and N_0; note $q \upharpoonright N_1 \neq r \upharpoonright N_1$. By the universality of M_1, there is $f : N_1 \to_{M_0} M_1$. By the nonsplitting of q and r over M_0, we get

$$\begin{aligned} f(q \upharpoonright N_1) &= q \upharpoonright f(N_1) \\ f(r \upharpoonright N_1) &= r \upharpoonright f(N_1) \end{aligned}$$

These were constructed to be not equal, but both are equal to $p \upharpoonright f(N_1)$, a contradiction.

$$\begin{tikzcd}
& N & \\
& & M_1 \arrow[ul] \\
N_1 \arrow[uur] \arrow[rr, "f"] & & f(N_1) \arrow[u] \\
& M_0 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

†

Before moving on to extension, we give some applications of nonsplitting to the stability spectrum problem. See [Vasb, Corollary 11.4] for a result in the other direction.

Theorem 5.8 ([GV06b, Theorem 0.2])**.** *Assume $\mathcal{K}$ has amalgamation. If $\mathcal{K}$ is μ-Galois stable and μ-tame, then $\mathcal{K}$ is κ^μ-Galois stable for every κ.*

Proof: Fix $M_0 \in \mathcal{K}_\kappa$. Using amalgamation, build $M_0 \prec M \in \mathcal{K}_{\kappa^\mu}$ such that, for every $N_0 \prec M$ of size μ, M is μ-universal over N_0. If $p \in \mathrm{gS}(M)$, then there is $N_p \prec M$ of size μ such that p does not μ-Galois split over N_p by stability and Theorem 5.5. Now find $N_p \prec N'_p \prec M$ of size μ such that N'_p is μ-universal over N_p. Then the map $p \in \mathrm{gS}(M) \mapsto (N'_p, p \upharpoonright N'_p)$ is injective by Theorem 5.7 and it's image has size $\kappa^\mu \cdot \mu = \kappa^\mu$ by assumption. †

Theorem 5.9 ([BKV06, Theorem 3.5])**.** *Assume $\mathcal{K}$ has amalgamation. If $\mathcal{K}$ is weakly (κ, κ^+)-tame, then κ-Galois stability implies κ^+-Galois stability.*

This theorem really uses a form of weak locality, but this is equivalent to tameness.

Proof: Suppose not. Then there is $M^* \in \mathcal{K}_{\kappa^+}$ with $\geq \kappa^{++}$-many types over it. By Proposition 3.16.(2), we can extend M^* to a saturated model, so we may assume it is saturated. Write M^* as the union of a resolution $\langle M_i \mid i < \kappa^+ \rangle$ with M_{i+1} κ-universal over M_i.

Claim: There is some $i_* < \kappa^+$ such that there are $\geq \kappa^{++}$-many types over M^* that do not κ-Galois split over M_{i_*}.

For each of the κ^{++} many types, by existence (Theorem 5.5) and monotonicity (Exercise 5.3.(2) there is some $i < \kappa^+$ such that it doesn't κ-Galois split over M_i. By the pigeonhole principle, some i_* occurs κ^{++}-many times. $\dagger_{\text{Claim}}$

For $p \in \mathrm{gS}(M_i)$, say that p *has many extensions* iff it has $\geq \kappa^{++}$-many extensions to M^* that don't κ-Galois split over M_{i_*}. By monotonicity, if $i > \kappa$ and $p \in \mathrm{gS}(M_i)$ has many extensions, then it doesn't κ-Galois split over M_{i_*}. Note each of the following:

(1) For each $i > i_*$, there is $p \in \mathrm{gS}(M_i)$ with many extensions.
(2) For each $j > i > i_*$, if $p \in \mathrm{gS}(M_i)$ has many extensions, then it can be extended to at least one $q \in \mathrm{gS}(M_j)$ with many extensions.
(3) For each $j > i > i_*$, if $p \in \mathrm{gS}(M_i)$ has many extensions, then it can be extended to exactly one $q \in \mathrm{gS}(M_j)$ with many extensions.

(1) and (2) are straightforward by counting. To see (3), if $q^1, q^2 \in \mathrm{gS}(M_j)$ have many extensions, then both don't κ-Galois split over M_{i_*}. Since M_i is κ-universal over M_{i_*} (and $\mathcal{K}$ is (κ, κ)-tame), then uniqueness (Theorem 5.7) implies that $q^1 = q^2$.

But now we can arrive at a contradiction. Fix some $p_0 \in \mathrm{gS}(M_i)$ with many extensions by (1). Each type in $p \in \mathrm{gS}(M_*)$ extending p_0 that doesn't κ-Galois split over M_{i_*} satisfies one of two possibilities:

(a) $p \upharpoonright M_i$ has many extensions for every $i < \kappa^+$; or
(b) there is some $i_p < \kappa^+$ such that $p \upharpoonright M_{i_p}$ does not have many extensions.

By κ-Galois stability, there are at most $\kappa \cdot \kappa^+$ many Galois types satisfying (b). If $q, r \in \mathrm{gS}(M_*)$ satisfy (a), then $q \upharpoonright M_i = r \upharpoonright M_i$ for all i by (3). By the weak tameness, $q = r$. Thus there are at most κ^+ many types that don't κ-Galois split over M_{i_*}, a contradiction. †

This can be extended to the following result. Note this extends a theorem of Morley from first-order!

Exercise 5.10 ([BKV06, Corollary 2.6]). *Suppose that $\mathcal{K}$ has $LS(\mathcal{K}) = \omega$, amalgamation, and is $< \omega$-tame. Then ω-Galois stability implies λ-Galois stability for all λ.*

Extension: We will prove several versions of this.

Theorem 5.11 (Extension, version 1, [Bal09, Lemma 12.6]). *If $M_0 \prec M_1 \prec M_2$ from $\mathcal{K}_\mu$, M_1 is μ-universal over M_0, and $p \in gS(M_1)$ does not μ-Galois split over M_0, then there is an extension of p to M_2 that does not μ-Galois split over M_0.*

Proof: By universality, find $f : M_2 \to_{M_0} M_1$. Extend this to an isomorphism $f_* : M_3 \cong M_1$ for some $M_3 \succ M_2$. Set $q = f_*^{-1}(p) \in \mathrm{gS}(M_3)$. Since p does not μ-Galois split over M_0, invariance implies that q does not μ-Galois split over M_0. Then, by this nonsplitting applied to the below diagram, $q \upharpoonright M_1 = f_* (q \upharpoonright M_3) = p$.

$$\begin{tikzcd}
& M_3 & \\
M_3 \arrow[ur] \arrow[rr, "f_*", "\cong"'] & & M_1 \arrow[ul] \\
M_2 \arrow[u] \arrow[urr, "f"'] & & \\
M_1 \arrow[u] & & \\
& M_0 \arrow[ul] \arrow[uuur] &
\end{tikzcd}$$

Thus, $q \upharpoonright M_2$ is the desired nonsplitting extension of p. †

We wish to prove another version, Theorem 5.25 and Corollary **??**. To do so, we introduce the notion of brimful models. In what follows, we're interested in looking at two AECs: the fixed AEC $\mathcal{K}$ that we work with and the AEC $(LO, \subset)$ of all linear orders. Note that brimful linear orders are designed to function outside of amalgamation, so we don't assume this. In this way, they are similar to the existentially closed models of Theorem 4.9.

Definition 5.12.

(1) *Let $M_1 \prec M_2 \prec M_3$. We say M_2 is* κ-universal over M_1 in M_3 *iff for all $M_1 \prec N \prec M_3$ with $\|N\| \leq \kappa$, there is $f : N \to_{M_1} M_2$.*

(2) *M is* κ-brimful *iff for all $M_1 \prec M$ with $\|M_1\| < \kappa$, there is $M_1 \prec M_2 \prec M$ such that*
   (a) *$\|M_1\| = \|M_2\|$; and*
   (b) *M_2 is $\|M_1\|$-universal over M_1 in M.*
   M is brimful *iff it is $\|M\|$-brimful.*

The first definition is some relativized notion of model homogeneity (Definition 3.14) that restricts consideration to some ambient model. This is good in many cases: if $\mathcal{K}$ is not κ-Galois stable, then we couldn't build such an unrestricted model in size κ. However, we brimfulness, we can restrict to a model that doesn't realize too many types and build the covers that we want.

We quickly turn to the question of building brimful linear orders, as the next result says that brimful linear order give rise to brimful EM models.

Proposition 5.13. *Let $\Phi \in \Upsilon[\mathcal{K}]$. If I is κ-brimful as a linear order, then $EM_\tau(I, \Phi)$ is κ-brimful as a member of $\mathcal{K}$.*

Proof: The moral of this proof is to grow everything to be of the form $EM_\tau(A, \Phi)$.

Let $M_1 \prec EM_\tau(I, \Phi)$ of size $< \kappa$. Then there is $I_1 \subset I$ of size $< \kappa$ such that $M_1 \prec EM_\tau(I_1, \Phi)$. Find the 'brimful cover' I_2 of I_1 in I. We claim that $EM_\tau(I_2, \Phi)$ is the brimful cover of M_1 in $EM_\tau(I, \Phi)$. Let $M_1 \prec N \prec EM_\tau(I, \Phi)$ of size $\|M_1\|$. We can find $I_1 \subset I_* \subset I$ of size $\|M_1\|$ such that $N \prec EM_\tau(I_*, \Phi)$. Passing to the linear orders, there is $f : I_* \to_{I_1} I_2$. This lifts to $\hat{f} : EM_\tau(I_*, \Phi) \to_{EM_\tau(I_1, \Phi)} EM_\tau(I_2, \Phi)$. Then this restricts to

$$\hat{f} \upharpoonright N : N \to_{M_1} EM_\tau(I_2, \Phi)$$

†

We know that EM models built on ordinals don't realize too many types from Theorem 3.37, so this seems like a good candidate. However, they are also very rigid, which is bad. We don't have to travel too far afield, though, and EM models build on[12] ${}^{<\omega}\lambda$ serve our purposes.

Fixing an ordinal γ, the set ${}^{<\omega}\gamma$ consists of the finite sequences of ordinals less than γ (conceived of as functions from their length to γ). We often partially order this as a tree by initial segment. However, here we will linearly order it lexicographically by

$$\eta <_{lex} \nu \iff \begin{cases} \ell(\eta) = n \\ \text{or} \\ \eta(n) < \nu(n) \end{cases} \qquad n := \ell(\eta \cap \nu) = \max\{k < \omega \mid \eta \upharpoonright k = \nu \upharpoonright k\}$$

Exercise 5.14. *$({}^{<\omega}\gamma, <_{lex})$ is a linear order.*

[12] In class, I wrote this a '$\lambda^{<\omega}$.' However, this ordering is more common.

For this bit, we might forget to write '*lex*,' but always mean it if we are treating ${}^{<\omega}\gamma$ as a linear order. Before proving they are brimful, we prove two purely order-theoretic results about them.

Proposition 5.15.

(1) ${}^{<\omega}\gamma$ *is not well-founded for* $\gamma > 1$*, but it has no uncountable decreasing sequences.*
(2) *Given* $\theta \leq \gamma < \theta^+$*, we can embed* ${}^{<\omega}\gamma$ *into* ${}^{<\omega}\theta$.

Proof: (2) actually implies that ${}^{\omega}\gamma$ is not well founded (as well-founded orders don't contain copies of extensions of themselves), but it's worth writing down an explicit example. Set $a_n \in {}^{<\omega}\gamma$ by $a_0 = \langle 1 \rangle$ and $a_{n+1} = \langle 0 \rangle^\frown a_n$. Then $a_{n+1} <_{lex} a_n$.

For the second part of (1), suppose for contradiction that $\langle \eta_\alpha \in {}^{<\omega\gamma} \mid \alpha < \omega_1 \rangle$ was decreasing. Since $\ell(\eta_\alpha)$ takes value in a countable set, there is uncountable $X \subset \omega_1$ such that $\alpha \in X \mapsto \ell(\eta_\alpha)$ is constant. Now color pairs from X with the first place they differ. By Infinite Ramsey's Theorem, there is an infinite homogeneous $Y \subset X$ with color $n_* < \omega$. For $\alpha, \beta \in Y$, $\eta_\alpha <_{lex} \eta_\beta$ iff $\eta_\alpha(n_*) < \eta_\beta(n_*)$. But since Y is infinite and $\langle \eta_\alpha \mid \alpha \in Y \rangle$ is decreasing, this gives an infinite decreasing sequence of ordinals, a contradiction.

For (2), we first do the following by induction:

Claim: For $\gamma < \theta^+$, we can find an embedding $f_\gamma : \gamma \to {}^{<\omega}\theta$.

For $\gamma = \theta$ (or even $\leq \theta$), this is obvious. For $\gamma' = \gamma + 1$, set

$$f_{\gamma'}(i) = \begin{cases} \langle 0 \rangle^\frown f_\gamma(i) & i < \gamma \\ \langle 1 \rangle & i = \gamma \end{cases}$$

For γ limit, we know $\operatorname{cf} \gamma \leq \theta$, so we can write $\gamma = \sup_{j<\delta} \gamma_j$ for $\langle \gamma_j \rangle$ increasing, continuous and $\delta = \operatorname{cf} \delta \leq \theta$. Then set

$$f_\gamma(i) = \langle \beta \rangle^\frown f_\beta(i) \text{ when } i \in [\gamma_\beta, \gamma_{\beta+1})$$

This works. $\dagger_{\text{Claim}}$

Now we can define $g_\gamma : {}^{<\omega}\gamma \to {}^{<\omega}\theta$ by

$$\eta = \langle \eta(0), \ldots, \eta(n-1) \rangle \in {}^{<\omega}\gamma \mapsto f_\gamma(\eta(0))^\frown \ldots {}^\frown f_\gamma(\eta(n-1))$$

$\dagger$

Lemma 5.16 ([KS96, Lemma 3.7])**.** *For each* λ*,* ${}^{<\omega}\lambda$ *is brimful.*

For this proof, recall that, given $I \subset J$, cuts in I determined by J are the same as quantifier-free types over J that are realized in I.

Proof: Let $J \subset {}^{<\omega}\lambda$. Find $A \subset \lambda$ of size $|J|$ such that $J \subset {}^{<\omega}A$ and set $\theta = |A|$.

Claim: There are θ-many quantifier-free types over ${}^{<\omega}A$ realized in ${}^{<\omega}\lambda$.

For $\eta \in {}^{<\omega}\lambda$, set $n_\eta = \min\{n \mid tp_{qf}(\eta/{}^{<\omega}A; {}^{<\omega}\lambda) = tp_{qf}(\eta \upharpoonright n/{}^{<\omega}A; {}^{<\omega}\lambda)\}$; clearly $\ell(\eta)$ is such an n, so this set is nonempty. Then $tp_{qf}(\eta/{}^{<\omega}A; {}^{<\omega}\lambda)$ is totally determined by

- $\eta \upharpoonright (n_\eta - 1) \in {}^{<\omega}A$ (which is of size θ); and
- $tp_{qf}(\eta(n_\eta)/A; \lambda)$ (of which there are θ-many).

Since $\theta \times \theta = \theta$, we are done. $\dagger_{\text{Claim}}$

Define ${}^{<\omega}A \subset J^* \subset \theta$ as follows.

Idea: We want to fill in all the cuts with a copy of ${}^{<\omega}\theta$ or as near as possible. Then this gives the brimful cover since any extension in ${}^{<}\omega\lambda$ looks like an extension by ${}^{<\omega}\gamma$ for some $\gamma < \theta^+$. Then apply Proposition 5.15.(2).

Construction: Set Γ to be the cuts of ${}^{<\omega}\lambda$ over ${}^{<\omega}A$; formally, these are the quantifier-free types over ${}^{<\omega}A$ realized in ${}^{<\omega}\lambda$. For each p, find the following:

- $\eta_p = \eta \upharpoonright (n_\eta - 1) \in {}^{<\omega}A$ for some/any $\eta \vDash p$;
- α_p is the supremum of all α and β_p is the infimum of all β such that
$$`\eta_p^\frown\langle\alpha\rangle < x < \eta_p^\frown\langle\beta\rangle' \in p$$
If these are empty, then $\alpha_p = -\infty$ or $\beta_p = \infty$ (depending); note that at most one is empty;
- γ_p^* is the unique ordinal so $\alpha_p + \gamma_p = \beta_p$ and
$$\gamma_p = \begin{cases} \beta_p & \text{if } \alpha_p = -\infty \\ \theta & \text{if } \beta_p = \infty \\ \min\{\gamma_p^*, \theta\} & \text{o/w} \end{cases}$$

Now set
$$J^* = {}^{<\omega} \cup \{\eta_p^\frown\langle\alpha_p + \delta\rangle^\frown {}^{<\omega}\theta \mid \delta < \gamma_p\}$$
with the convention that $-\infty + \delta = \delta$.

This works:

Exercise 5.17. *Prove this!*

†

Corollary 5.18. *If $\Phi \in \Upsilon[\mathcal{K}]$, then $EM_\tau({}^{<\omega}\lambda, \Phi)$ is brimful.*

There are several variations of types we would like to consider.

Definition 5.19. *Let $M \in \mathcal{K}$ and $p \in gS(M)$.*

(1) *p is* nonalgebraic *iff p is not realized in M.*
(2) *p is* big *iff there is a model with $> \|M\|$ realizations of p.*
(3) *p is* minimal *iff p is big and there is exactly one big extension to any $\|M\|$-sized model.*

We use gS^{na}, gS^{big}, and gS^{min} to refer to the corresponding set of Galois types.

Minimal types, if you can find them, are very nice. [She01] uses them to build a very nice independence notion called a good λ-frame (see Definition **??**).

Exercise 5.20. (1) *Show that minimal implies big implies nonalgebraic.*
(2) *Assume amalgamation. p is big iff it has at least one nonalgebraic extension to any $\|M\|$-sized model. It also has a big extension to an $\|M\|$-sized model.*

In first order, big and algebraic coincide. However, the following simple example shows that this is not the case in a general AEC.

Example 5.21 (Kueker)**.** *Set τ to consist of a single unary predicate P and $\mathcal{K}$ consist of all τ-structures M with $|P^M| \leq 1$ with $\prec_\mathcal{K} = \subset$. Then, if $M \prec N$ with $a \in P^N - P^M$, we have $gtp(a/M; N)$ nonalgebraic, but with no model realizing it more than once.*

We need much less than compactness to show that big and algebraic are the same. Instead, it only requires the following strengthening of amalgamation.

Definition 5.22 (Disjoint Amalgamation)**.** *We say that $\mathcal{K}$ has the* disjoint amalgamation property (DAP) *iff for every $M \prec N_0, N_1$ from $\mathcal{K}$, there is an amalgam $N^* \succ N_0$ and $f : N_1 \to_M N^*$ such that $N_0 \cap f(N_1) = M$. We can use the different naming properties from amalgamation (Definition 3.7) here as well.*

Exercise 5.23 (e. g., [Hod93, 6.4.3])**.** *For any first-order theory T, $(Mod\, T, \prec)$ has disjoint amalgamation.*

Proposition 5.24. *If $\mathcal{K}$ has λ-disjoint amalgamation, then any nonalgebraic type over a λ-sized model is big.*

Proof: We use the criterion from Exercise 5.20. Let $p \in \mathrm{gS}^{na}(M)$ and $N \succ M$ of size λ. $p = \mathrm{gtp}(a/M; M^*)$, so find a disjoint amalgam of M^* and N over M

$$\begin{tikzcd} M^* \arrow[r, "f"] & N^* \\ M \arrow[u] \arrow[r] & N \arrow[u] \end{tikzcd}$$

Then $\mathrm{gtp}\,(f(a)/N; N^*)$ extends p and, by disjointness, is nonalgebraic. †

Now we can prove a second, more technical version of extension of nonsplitting types.

Theorem 5.25 (Extension, version 2, [Bal09, Lemma 12.13])**.** *Suppose $\mathcal{K}$ has amalgamation and $\Phi \in \Upsilon[\mathcal{K}]$. Let*

(1) $|I_0| \leq \kappa \leq \mu < \lambda$;
(2) $p \in gS^{na}\left(EM_\tau({}^{<\omega}\mu, \Phi)\right)$;
(3) $I_0 \subset {}^{<\omega}\mu$; *and*
(4) *p does not κ-Galois split over $EM_\tau(I_0, \Phi)$ and is realized in $EM_\tau({}^{<\omega}\lambda, \Phi)$.*

Then there is some $q \in gS^{na}\left(EM_\tau({}^{<\omega}\lambda, \Phi)\right)$ and $I_0 \subset I_1 \subset {}^{<\omega}\mu$ such that

(1) *q extends p and does not κ-Galois split over $EM_\tau(I_1, \Phi)$; and*
(2) *$I_1 - I_0$ is finite.*

As with Theorem 4.29, this is a technical result about the behavior of EM models. To apply it, we will use the same trick where categoricity allows us to recognize EM models as saturated.

Corollary 5.26. *Suppose $\mathcal{K}$ has amalgamation and arbitrarily large models and is categorical in $\mathcal{K}$. If $M \in \mathcal{K}_{(LS(\mathcal{K}),\lambda)}$ is Galois saturated, $M \prec N \in \mathcal{K}_\lambda$, and $p \in gS^{na}(M)$, then there are $q \in gS^{na}(N)$ and $N_q \prec M$ such that $p \leq q$ and q does not $LS(\mathcal{K})$-Galois split over N_q.*

Proof: Let $\Phi \in \Upsilon[\mathcal{K}]$. By categoricity, we have that $N \cong EM_\tau({}^{<\omega}\lambda, \Phi)$ and that this model is Galois saturated by Fact 4.32. Set $\mu = \|M\|$. Then by Corollary 4.34, $EM_\tau({}^{<\omega}\mu, \Phi)$ is Galois saturated and, thus, isomorphic to M. By Theorem 3.17, N is model homogeneous. This means that we can find an automorphism of N that sends M to $EM_\tau({}^{<\omega}\mu, \Phi)$. So, WLOG, set $N = EM_\tau({}^{<\omega}\lambda, \Phi)$ and $M = EM_\tau({}^{<\omega}\mu, \Phi)$.

By Theorem 3.37 and categoricity, $\mathcal{K}$ is LS($\mathcal{K}$)-Galois stable. By Theorem 5.5, there is some $N_p \prec M$ such that p does not LS($\mathcal{K}$)-Galois split over N_p. Now Theorem 5.25 provides the desired $q \in \mathrm{gS}^{na}(N)$ and $N_p \prec N_q \prec M$. †

Proof of 5.25: We know p is realized in an EM model, so we have it realized by $\sigma(\mathbf{a})$ for some $\mathbf{a} \in {}^{<\omega}\lambda$. Set $I_1 := I_0 \cup (\mathbf{a} \cap {}^{<\omega}\mu)$.

Define $J' := {}^{<\omega}\lambda \cup \{\mathbf{a}'\}$ to be ordered so that a'_i is the maximal element of the cut over ${}^{<\omega}\mu$ that contains a_i. Explicitly, this is

(1) $tp_{qf}(\mathbf{a}/{}^{<\omega}\mu; {}^{<\omega}\lambda) = tp_{qf}(\mathbf{a}'/{}^{<\omega}\mu; {}^{<\omega}\lambda)$; and
(2) for all a'_i and $s \in {}^{<\omega}\lambda$, **if** for all $t \in {}^{<\omega}\mu$, $t < a'_i$ iff $t < s$, **then** $s < a'_i$.

By the first point we can extend J' to J'' with $f \in \mathrm{Aut}_{{}^{<\omega}\mu} J''$ such that $f(\mathbf{a}) = \mathbf{a}'$. Then $\sigma(\mathbf{a}') \vDash p$. Set

$$q = \mathrm{gtp}\left(\sigma(\mathbf{a}')/EM_\tau({}^{<\omega}\lambda, \Phi); EM_\tau(J, \Phi)\right)$$

Claim: q does not κ-Galois split over $EM_\tau(I_1, \Phi)$.

Suppose that we had a set-up to check for splitting

$$\begin{tikzcd}
& EM_\tau({}^{<\omega}\lambda, \Phi) & \\
N_1 \arrow[ur] \arrow[rr, "h", "\cong"'] & & N_2 \arrow[ul] \\
& EM_\tau(I_1, \Phi) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Find $I_1 \subset L \subset \lambda^{<\omega}$ of size κ such that $EM_\tau(L, \Phi)$ contains N_1 and N_2.

Subclaim: There is $g : L\mathbf{a}' \to_{I_1} ({}^{<\omega}\mu)\mathbf{a}'$ that fixes $\mathbf{a}'$.

We know that every decreasing sequence in ${}^{<\omega}\mu$ is countable, so expand I_1 to add, for each a'_i, a countable decreasing sequence from ${}^{<\omega}\mu$ with a'_i as its infimum. By brimfulness Lemma 5.16, there is an embedding $L \to_{I_1} {}^{<\omega}\mu$. By construction, adding $\mathrm{id}_{\mathbf{a}'}$ to this embedding is our g. $\dagger_{\text{Subclaim}}$

This induces $\hat{g}$ and the following diagram.

$$\begin{tikzcd}
& & & & EM_\tau\left({}^{<\omega}\mu\mathbf{a}', \Phi\right) & \\
& EM_\tau(L\mathbf{a}', \Phi) & & & EM_\tau\left(g(L)\mathbf{a}', \Phi\right) \arrow[u] & \\
& EM_\tau(L, \Phi) \arrow[u] & & & EM_\tau\left(g(L), \Phi\right) \arrow[u] & \\
N_1 \arrow[ur] \arrow[rr, "h", "\cong"'] & & N_2 \arrow[ul] \arrow[r, Rightarrow, "\hat{g}"] & \hat{g}(N_1) \arrow[ur] \arrow[rr, "\hat{g}\circ h\circ \hat{g}^{-1}", "\cong"'] & & \hat{g}(N_2) \arrow[ul] \\
& EM_\tau(I_1, \Phi) \arrow[ul] \arrow[ur] & & & EM_\tau(I_1, \Phi) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

We know that $q \restriction EM_\tau({}^{<\omega}\mu, \Phi)$ is p and, therefore, does not κ-Galois split over $EM_\tau(I_1, \Phi)$ (we've used monotonicity of nonsplitting (Exercise 5.3.(2)) here as well).

Before moving on, note that

$$q \restriction \hat{g}(N_\ell) \quad = \quad \mathrm{gtp}\left(\sigma(\mathbf{a}')/\hat{g}(N_\ell); EM_\tau\left(g(K)\mathbf{a}', \Phi\right)\right)$$

So,

$$\begin{aligned}
\hat{g}^{-1}\left(q \restriction \hat{g}(N_\ell)\right) &= \mathrm{gtp}\left(\sigma(\mathbf{a}')/N_\ell; EM_\tau\left(K\mathbf{a}', \Phi\right)\right) \\
&= q \restriction N_\ell
\end{aligned}$$

Then we compute the following (the first is the nonsplitting of p):

$$\begin{aligned} \hat{g} \circ h \circ \hat{g}^{-1} (q \upharpoonright \hat{g}(N_1)) &= q \upharpoonright \hat{g}(N_2) \\ h \circ \hat{g}^{-1} (q \upharpoonright \hat{g}(N_1)) &= \hat{g}^{-1} (q \upharpoonright \hat{g}(N_2)) \\ h(q \upharpoonright N_1) &= q \upharpoonright N_2 \end{aligned}$$

Thus, we have our Claim and Theorem proved. $\dagger_{\text{Claim, Theorem}}$

5.2. **Getting minimal types.** We want to show that we have a plentiful supply of minimal types. The following is our goal:

Goal (Theorem 5.34)**.** *Suppose $\mathcal{K}$ has amalgamation, is categorical in $\lambda > LS(\mathcal{K})$, and $\Upsilon[\mathcal{K}] \neq \emptyset$. If $M \in \mathcal{K}$ is Galois saturated with $LS(\mathcal{K}) < \|M\| < \lambda$, then there is a minimal Galois type over M.*

The outline of this theorem is fairly straightforward. If there are no minimal types over saturated models, then we want to build a tree of types indexed by $2^{<\log \|M\|}$. We build them as a coherent system to ensure that there is a union type along each branch and this contradicts stability as in Theorem 5.5. Saturation is used to get isomorphisms. We have two issues at the limit stage:

- **Ensuring the union model is saturated.** This is taken care of by using limit models (Definition 5.27) and showing that they will always be Galois saturated under these hypotheses (Theorem 5.31).
- **Big types might turn into merely nonalgebraic types.** This will make our induction step impossible. We take care of this by showing that, under our hypotheses, big and nonalgebraic types coincide (Theorem 5.33)!

Limit models were introduced in [KS96,She01] and have been variously called (μ, α)-saturated [KS96, She01] or brimmed [She09a]. [SV99] introduced the terminology of limit models, which is now standard. One use is to give a way of talking about Galois saturated models in the size LS($\mathcal{K}$) or more generally if you're doing a local analysis of $\mathcal{K}_\lambda$. Since then, they've turned into a nice way to measure notions related to superstability, especially around the uniqueness of limit models.

Definition 5.27.
(1) *M is* κ-universal over N *iff for every $N' \succ N$ of size κ, there is $f : N' \to_N M$.*
M is universal over N *iff it is $\|N\|$-universal over N.*
(2) *For $\alpha < \mu^+$ limit, we say that M is a* (μ, α)-limit model over N *iff there is an increasing, continuous sequence $\langle M_i \in \mathcal{K}_\mu \mid i < \alpha \rangle$ such that*
  (a) *$N = M_0$; and*
  (b) *M_{i+1} is universal over M_i.*
(3) *$\mathcal{K}_\mu$ has* unique limit models *iff for all $\alpha, \beta < \mu^+$, all (μ, α)-limits and (μ, β)-limits are isomorphic.*

The first part is a straightforward back and forth and exemplifies that limit models really reduce to regular length. The second part relates limit models to Galois saturation.

Exercise 5.28.

(1) *If M_ℓ is a (μ, α_ℓ)-limit model and $cf\, \alpha_1 = cf\, \alpha_2$, then $M_1 \cong M_2$. If M_ℓ are both these limits over a common N, then this isomorphism can be chosen to fix it.*
(2) *If (μ, μ)-limits are $cf\, \mu$-Galois saturated.*

This tells us that, to show (μ, α)-limit models are saturated, it's enough to show some (μ, α)-limit model is saturated. So we build them with EM models.

Lemma 5.29. *Suppose that $\mathcal{K}$ has amalgamation, is λ-categorical, $\Upsilon[\mathcal{K}] \neq \emptyset$, and $\alpha < \lambda^+$. Let I be a linear order with the following property:*

$$(*) \qquad \text{For any } \alpha < |I|^+, \text{ there is } f : \alpha \times I \to I$$

Then for each $\alpha < \mu^+$, $EM_\tau((\alpha+1) \times I, \Phi)$ is universal over $EM_\tau(\alpha \times I, \Phi)$.

Proof: By categoricity, Fact 4.32, and Theorem 3.17, we know that $EM_\tau(\lambda \times I, \Phi)$ is Galois saturated and model homogeneous. Let $M \succ EM_\tau(\alpha \times I, \Phi)$. By model homogeneity, we can embed this into the λ-sized model. In particular, we can find $Y \subset \lambda - (\alpha + 1)$ of size $|I|$ and

$$f : M \to_{EM_\tau(\alpha \times I, \Phi)} EM_\tau((\alpha \cup Y) \times I, \Phi)$$

Then $\mathrm{otp}(Y) < |I|^+$, so by $(*)$, there is some $g : Y \times I \to I$. Then, $\mathrm{id}_{\alpha \times I} \cup g$ extends to

$$\hat{g} : EM_\tau((\alpha \cup Y) \times I, \Phi) \to_{EM_\tau(\alpha \times I, \Phi)} EM_\tau((\alpha+1) \times I, \Phi)$$

Then $\hat{g} \circ f$ is the desired map. †

Corollary 5.30. *$EM_\tau(\alpha \times {}^{<\omega}\mu, \Phi)$ is (μ, α)-limit witnessed by the chain $\langle EM_\tau(\beta \times {}^{<\omega}\mu, \Phi) \mid \beta < \alpha \rangle$.*

Proof: By Lemma 5.29, it suffices to show that ${}^{<\omega}\mu$ satisfies $(*)$ from there. This follows by Proposition 5.15.(2): $\alpha \times {}^{<\omega}\mu$ naturally embeds into ${}^{<\omega}\max\{\alpha, \mu\}$ by $(\beta, \eta) \mapsto \langle \beta \rangle^\frown \eta$. †

Theorem 5.31. *Suppose that $\mathcal{K}$ has amalgamation, is λ-categorical, $\Upsilon[\mathcal{K}] \neq \emptyset$, and $\alpha < \mu^+ \leq \lambda$. Every (μ, α)-limit model is saturated.*

Proof: By Corollary 5.30, $EM(\alpha \times {}^{<\omega}\mu, \Phi)$ is (μ, α)-limit. By Corollary 4.34, this model is saturated. By Exercise 5.28.(1) (applied with $\alpha_1 = \alpha_2 = \alpha$), every (μ, α)-limit is Galois saturated. †

Now we wish to show that nonalgebraic types over saturated models are big. Note by Proposition 5.24, we would already know this if $\mathcal{K}$ had disjoint amalgamation. The following lemma would be enough with weak tameness; this allows us to use the uniqueness of nonsplitting extensions (Theorem 5.7).

Lemma 5.32. *Suppose that $\mathcal{K}$ has amalgamation, is χ-Galois stable, and is weakly (χ, μ)-tame for $LS(\mathcal{K}) \leq \chi < \mu$. If M is Galois saturated of size μ, $p \in gS^{na}(M)$ does not χ-Galois split over M_p, and $N \succ M$ is of size μ, then p has a nonalgebraic extension to N.*

Proof: By χ-stability and saturation, find $M_p \prec M_* \prec M$ of size χ that is universal over M_p. Let $N' \succ N$ be a Galois saturated model of size μ. Then there is $f : M \cong_{M_p} N'$. Both p and $f(p) \restriction M$ are extensions of $p \restriction M_*$ that do not χ-Galois split over $p \restriction M_p$ by invariance. By the uniqueness of nonsplitting extensions, this means that $p = f(p) \restriction M$. Then $f(p) \restriction N$ is the desired type; it is nonalgebraic because $f(p)$ is, which in turn is nonalgebraic because p is. †

However, we would prefer to work without tameness when we can. The following argument works a little harder to eliminate this. The key is to find the $M_p \prec M_* \prec M$ as above *without* assuming that there is any separation between χ and μ (and thus appealing to the (μ, μ)-tameness of every AEC).

Theorem 5.33. *Suppose that $\mathcal{K}$ has amalgamation, is categorical in λ, and $\Upsilon[\mathcal{K}] \neq \emptyset$. If M is Galois saturated and $LS(\mathcal{K}) < \|M\| < \lambda$, then any nonalgebraic type over M is big.*

Proof: Let $p \in \mathrm{gS}^{na}(M)$ and $M \prec N \in \mathcal{K}_{\|M\|}$. We want to show that p has a nonalgebraic extension to N. Since there are no maximal models, we can find $N \prec N^* \in \mathcal{K}_\lambda$. By Corollary **??**, there is a nonalgebraic extension q of p to N^*. Then $q \restriction N$ is as desired. †

We are now ready to prove our goal:

Theorem 5.34 ([Bal09, Theorem 12.23])**.** *Suppose $\mathcal{K}$ has amalgamation, is categorical in $\lambda > LS(\mathcal{K})$, and $\Upsilon[\mathcal{K}] \neq \emptyset$. If $M \in \mathcal{K}$ is Galois saturated with $LS(\mathcal{K}) < \|M\| < \lambda$, then there is a minimal Galois type over M.*

Proof: Set $\kappa := \|M\|$. By Theorem 3.37, $\mathcal{K}$ is stable in κ. Find the least μ such that $2^\mu > \kappa$. Suppose there is no minimal type over a saturated model of size κ. Then there is no minimal type over any saturated model of size κ by Theorem 3.17. We are going to build a coherent tree of nonminimal types over saturated models.

Explicitly, we build $M_\eta, N_\eta, f_{\eta,\nu}, a_\eta$ for $\eta < \nu \in {}^{\leq\mu}2$ such that

1. $\langle M_\eta \in \mathcal{K}_\kappa \mid \eta \in {}^{\leq\mu}2\rangle$ is increasing continuous;
2. each M_η is Galois saturated with $M_{\eta^\frown\langle\ell\rangle}$ universal over M_η;
3. $(a_\eta, M_\eta, N_\eta) \in \mathcal{K}^3_\kappa$ has a big Galois type;
4. $f_{\eta,\nu} : N_\eta \to_{M_\eta} N_\nu$ such that
   (a) $f_{\eta,\nu}(a_\eta) = a_\nu$ and
   (b) if $\eta < \nu < \rho$, then $f_{\eta,\rho} = f_{\nu,\rho} \circ f_{\eta,\nu}$;
5. $M_{\eta^\frown\langle 0\rangle} = M_{\eta^\frown\langle 1\rangle}$ and $\mathrm{gtp}(a_{\eta^\frown\langle 0\rangle}/M_{\eta^\frown\langle 0\rangle}; N_{\eta^\frown\langle 0\rangle}) \neq \mathrm{gtp}(a_{\eta^\frown\langle 1\rangle}/M_{\eta^\frown\langle 1\rangle}; N_{\eta^\frown\langle 1\rangle})$; and
6. $\langle N_\eta, f_{\eta,\nu} \mid \eta < \mu \in {}^{\leq\mu}2\rangle$ is an increasing system of models from $\mathcal{K}_\kappa$.

This is enough: Find $M \in \mathcal{K}_\kappa$ to contain every M_η for $\eta \in {}^{<\mu}$. By continuity of the branches, M also contains every M_η for $\eta \in {}^\mu 2$. For each such η, $\mathrm{gtp}(a_\eta/M_\eta; N_\eta)$ extends to a type $p_\eta \in \mathrm{gS}(M)$ by amalgamation. By construction, the p_η are distinct for distinct η, so this contradicts stability.

Construction: As always, we go by induction on the length of η.

<u>$\eta = \emptyset$</u>: $M_\emptyset$ is an saturated model of size κ. $p_\emptyset \in \mathrm{gS}(M_\emptyset)$ is any big type, find a triple $(a_\emptyset, M_\emptyset, N_\emptyset)$ realizing it.

<u>$\eta = \mu^\frown\langle\ell\rangle$</u>: p_ν is big but not minimal, so it has two big extensions to some $M' \succ M_\nu$. Extend M' to some $M_{\nu^\frown\langle\ell\rangle}$ that is saturated and universal over M_ν (and the same for $\ell = 0, 1$), and extend the different big types to nonalgebraic $p_{\nu^\frown\langle 0\rangle}$ and $p_{\nu^\frown\langle 1\rangle}$. By Theorem 5.33, they are still big. Find triples $(a_{\nu^\frown\langle\ell\rangle}/M_{\nu^\frown\langle\ell\rangle}, N*_\ell)$ realizing $p_{\nu^\frown\langle\ell\rangle}$. By construction, a_ν and $a_{\nu^\frown\langle\ell\rangle}$ realize the same type over M_ν. Thus, we can find an amalgam

$$\begin{tikzcd}
N_{\nu^\frown\langle\ell\rangle} \arrow[rr, "g_\ell"] & & N_{\nu^\frown\langle\ell\rangle} \\
M_\nu \arrow[u] \arrow[r] & M_{\nu^\frown\langle\ell\rangle} \arrow[r] & N^*_\ell \arrow[u]
\end{tikzcd}$$

such that $g_\ell(a_\nu) = a_{\nu^\frown\langle\ell\rangle}$. Then we finish by setting, for $\rho \leq \nu$, $h_{\rho,\nu^\frown\langle\ell\rangle} = g_\ell \circ h_{\rho,\nu}$.

<u>η has limit length:</u> Form the directed limit of the systems so far:

$$\begin{aligned} M_\eta &= \bigcup_{\alpha<\ell(\eta)} M_{\eta\restriction\alpha} \\ (N_\eta, f_{\eta\restriction\gamma,\eta}) &= \lim_{\alpha<\beta<\ell(eta)} (N_{\eta\restriction\alpha}, f_{\eta\restriction\alpha,\eta\restriction\beta}) \\ a_\eta &= f_{\eta\restriction\alpha,\eta}(a_{\eta\restriction\alpha}) \end{aligned}$$

By construction, $a_\eta \notin M_\eta$, so its type is nonalgebraic. But we want more! By construction, M_η is $(\kappa, \ell(\eta))$-limit. By Theorem 5.31, it is Galois saturated. Thus, by Theorem 5.33, the type is big. So we may continue the construction. $\dagger_{\text{Construction, Theorem}}$

5.3. **Downward categoricity transfer.**

The categoricity transfers of Theorem 3.6 have two main pieces:

(1) If we start with categoricity in some λ^+ above the second Hanf number, then we can reflect categoricity down to the second Hanf number and weak tameness in that interval.
(2) If we start with categoricity in some λ, weak (λ, λ^+)-tameness, and a little more, then we can push categoricity to λ^+.

We've already seen the second part of (1) in Theorem 4.25. Now we do the first part. This involves not much about tameness, but a lot about Shelah's Presentation Theorem and EM models. The first approximation follows from the already proven Morley's Omitting Types Theorem 3.33.

Theorem 5.35. *If $\mathcal{K}$ has amalgamation and is categorical in $\lambda \geq \beth_{(2^\kappa)^+}$, then every $M \in \mathcal{K}_{\geq \beth_{(2^\kappa)^+}}$ is κ^+-Galois saturated.*

Proof: By Corollary 4.34, we know that the λ-sized models is Galois saturated. Suppose we have an M of sufficient size that is not κ^+-Galois saturated. Then, WLOG, $M \in \mathcal{K}_{\beth_{(2^\kappa)^+}}$ and there is $M_0 \prec M$ of size κ with $p \in \mathrm{gS}(M_0)$ that is not realized in M. Define the AEC $\mathcal{K}^{M_0}$ by

$$\begin{aligned} \tau^{M_0} &:= \tau(\mathcal{K}) \cup \{c_m \mid m \in M_0\} \\ \mathcal{K}^{M_0} &:= \{(N, a_m)_{m\in M_0} \mid N \in \mathcal{K} \text{ and } m \in M_0 \mapsto a_m \in N \text{ is a } \mathcal{K}\text{-embedding}\} \\ (N, a_m)_{m\in M_0} \prec_{\mathcal{K}^{M_0}} (N', b_m)_{m\in M_0} &\iff N \prec_{\mathcal{K}} N' \text{ and } a_m = b_m \end{aligned}$$

Exercise 5.36. *$\mathcal{K}^{M_0}$ is an AEC with $LS(\mathcal{K}^{M_0}) = \|M_0\| + LS(\mathcal{K})$.*

Then M has a natural expansion $(M, m)_{m\in M_0}$ to a model of $\mathcal{K}^{M_0}$ of size larger than its Hanf number. By Shelah's Presentation Theorem 3.25 and Morley's Omitting Types Theorem 3.33, there is $\Phi \in \Upsilon_\kappa[\mathcal{K}^{M_0}]$ that is patterned off $(M, m)_{m\in M_0}$ in the sense of Remark 3.34. This means that, after an application of Exercise 3.39, for every I and $i_1 < \cdots < i_n \in I$ and $\tau(\Phi)$-term σ, we have that $\mathrm{gtp}\,(\sigma(i_1, \ldots, i_n)/\emptyset; EM_{\tau^{M_0}}(I, \Phi))$ is realized in M. Since p is transferred into p^* which is a $\mathcal{K}^{M_0}$-Galois type over the empty set, this means that no $EM_\tau(I, \Phi)$ can be κ^+-Galois saturated. However, $EM_\tau(I, \Phi)$ is Galois saturated by categoricity, contradiction. $\dagger$

This result tells us that we can transfer categoricity between cardinals when we can 'catch our tail' of this operation. This involves finding a cardinal such that $\mu < \delta$ implies that $\beth_{(2^\mu)^+} < \delta$.

Corollary 5.37. *Suppose that $\mathcal{K}$ has amalgamation. If $\mathcal{K}$ is λ-categorical and $\lambda > \delta = \beth_\delta > LS(\mathcal{K})$, then $\mathcal{K}$ is δ-categorical.*

This is a nice result, but not quite good enough. To push categoricity down to the second Hanf number, we need something better than Morley's Omitting Types Theorem. The following is Shelah's Omitting Type Theorem. As we will see, a key difference is that this omitting types has some uniquely AEC parts to it. The proof given is copied from a note [Bonc] that I wrote after conversations with John Baldwin and Sebastien Vasey. There's also a much more in depth discussion there.

One nonstandard piece of notation:

Definition 5.38. *If $N \prec M$ and $p \in gS(N)$ with $\chi \leq \|N\|$, then we say that M omits p/E_χ iff for every $c \in M$, there is some $N^- \prec N$ of size $\prec \chi$ such that c does not realize $p \upharpoonright N^-$.*

Note that this is implied by omitting p and is the same under χ-tameness (or weak tameness if N is saturated). So we can think of this as a strong form of type omission. However, this is weaker than omitting the set $\{p \upharpoonright N^- : N^- \prec N \text{ and } \|N^-\| \leq \chi\}$ of restrictions of p. Each of the types in that set might be realized in M; however, there is no element of M that simultaneously realizes them all.

In the following all types are of length $< \omega$.

Theorem 5.39 (Shelah's Omitting Types Theorem)**.** *Let $\mathcal{K}$ be an AEC with $LS(\mathcal{K}) \leq \chi \leq \lambda$ with*

1. *$N_0 \prec N_1$ with $\|N_0\| \leq \chi$ and $\|N_1\| = \lambda$;*
2. *$\Gamma_0 = \{p_i^0 : i < i_0^*\}$ are Galois types over N_0; and*
3. *$\Gamma_1 = \{p_i^1 : i < i_1^*\}$ are Galois types over N_1 with $i_1^* \leq \chi$.*

Suppose that, for each $\alpha < (2^\chi)^+$, there is $M_\alpha \in \mathcal{K}$ such that

1. *$\|M_\alpha\| \geq \beth_\alpha(\lambda)$ and $N_1 \prec M_\alpha$;*
2. *M_α omits Γ_0; and*
3. *M_α omits p_i^1/E_χ for each $i < i_1^*$.*

Then we can find $\Phi \in \Upsilon_\chi[\mathcal{K}]$; increasing, continuous $\langle N_n' \in \mathcal{K}_{\leq \chi} : n \leq \omega \rangle$; and increasing Galois types $p_{i,n}^1 \in gS(N_n')$ for $n < \omega, i < i_1^*$ such that

1. *$N_0 = N_0' = EM_\tau(\emptyset, \Phi)$;*
2. *for each $n < \omega$, we have $N_n' \prec EM_\tau(n, \Phi)$ and $f_n : EM_\tau(n, \Phi) \to M_{\alpha_n}$ for some $\alpha_n < (2^\chi)^+$ such that $f_n(N_n') \prec N_1$.*
3. *$p_{i,n}^1 := f_n^{-1}(p_i^1 \upharpoonright f_n(N_n')) \in gS(N_n')$; and*
4. *for every infinite[13] I, $EM_\tau(I, \Phi)$ omits Γ_0 and omits any type that extends $\{p_{i,n}^1 : n < \omega\}$ in the following strong sense: if $p_{i,*}^1 \in gS(N_\omega')$ extends each $p_{i,n}^1$ and $J \subset I$ is of size $n < \omega$ with $a \in EM_\tau(J, \Phi)$, then a doesn't realize $p_{i,*}^1 \upharpoonright N_n' = p_{i,n}^1$.*

Proof: Stage 1 will build a language τ^+; it is essentially a language from Shelah's Presentation Theorem with some extra aspects tacked on. Stage 2 builds a "tree of indiscernibles." Stages 3 uses this tree to build the template Φ and finishes the proof.

Stage 1: Set $\tau^+ := \tau \cup \{F_n^i : i < \chi\}$, as in Shelah's Presentation Theorem. Let M be a τ structure such that $N_1 \prec M$ and M omits p_i^1/E_χ for each $i < i_1^*$. We describe a procedure to expand M to a τ^+-structure M^+ with certain properties: we want to define a cover $\{M_{\mathbf{a}} \in \mathcal{K} : \mathbf{a} \in M\}$ with the following properties:

1. If $\mathbf{a} \in N_0$, then $M_{\mathbf{a}} = N_0$ (so in particular, this is true for $\mathbf{a} = \emptyset$)
2. If $\mathbf{a} \in N_1$, then $M_{\mathbf{a}} \prec N_1$
3. For all $\mathbf{a}$, set $M_{\mathbf{a},1} := M_{\mathbf{a}} \cap N_1$. Then
$$N_0 \prec M_{\mathbf{a},1} \prec M_{\mathbf{a}}$$
4. For $i < i_1^*$, we have $p_i^1 \upharpoonright (M_{\mathbf{a},1})$ is omitted in $M_{\mathbf{a}}$.

[13]Note if I is not infinite, then N_ω' does not appear as a strong substructure.

We build this cover in ω many steps, building increasing covers $\{M^n_{\mathbf{a}} : n < \omega\}$ that get closer and closer.

$\mathbf{n = 0}$: Nothing special happens here. Start with $M^0_{\mathbf{a}} = N_0$ for all $\mathbf{a} \in N_0$. Then extend this to a cover of N_1, and then to a cover of M. Note we ignore conditions (3) and (4) here. Also, if $\mathbf{a} \in N_1$, we will not change $M^0_{\mathbf{a}}$ in the rest of the construction.

$\mathbf{2n+1}$: Suppose the increasing covers up to $2n$ are built. We take care of (3) in this step. First note that, for $\mathbf{a} \in N_1$, (3) is guaranteed, so no change should be done. This step is itself made up of ω many steps. Do the following construction by induction on the length of $\mathbf{a}$:
It might be the case that $\left(M^{2n}_{\mathbf{a}} \cup \bigcup_{\mathbf{b} \subsetneq \mathbf{a}} M^{2n+1}_{\mathbf{b}}\right) \cap N_1$ is not a τ-structure or in $\mathcal{K}$. However, we can find $N^{1,0} \prec N_1$ containing it of size χ. Then find $N^{2,0} \prec M$ containing $M^{2n}_{\mathbf{a}} \cup \bigcup_{\mathbf{b} \subsetneq \mathbf{a}} M^{2n+1}_{\mathbf{b}}$ of size χ. Iterate this process so

- $N^{1,i+1} \prec N_1$ contains $N^{2,i} \cap N_1$ and is of size χ; and
- $N^{2,i+1} \prec M$ contains $N^{1,i+1}$ and is of size χ.

In the end, set $M^{2n+1}_{\mathbf{a}} := \cup_{i<\omega} N^{2,i}$. Then we have $M^{2n+1}_{\mathbf{a}} \cap N_1 = \cup_{i<\omega} N^{1,i}$, which is a strong substructure of N_1, as desired. Also, since we included the $\bigcup_{\mathbf{b} \subsetneq \mathbf{a}} M^{2n+1}_{\mathbf{b}}$ term, this will form an increasing cover.

$\mathbf{2n+2}$: In this step, we take care of (4). Note that, by the odd step, $M^{2n+1}_{\mathbf{a},1} := M^{2n+1}_{\mathbf{a}} \cap N_1 \prec N_1$ is well defined. Again, we are going to expand our cover $\{M^{2n+1}_{\mathbf{a}} : \mathbf{a} \in M\}$ by induction on the length of $\mathbf{a}$:
Suppose that $M^{2n+2}_{\mathbf{b}}$ is defined for all proper subtuples $\mathbf{b}$ of $\mathbf{a}$. For each $i < i^*_1$, it might be the case that $m \in M^{2n+1}_{\mathbf{a}}$ realizes $p \upharpoonright M^{2n+1}_{\mathbf{a},1}$. For each such i and m, pick $M_{i,m} \prec M$ of size χ such that m does not realize $p \upharpoonright M_{i,m}$; such a model exists precisely because M omits p/E_χ. An important point is that $m \in N_1$ implies that $m \in M^{2n+1}_{\mathbf{a},1}$ and, therefore, already omits $p \upharpoonright M^{2n+1}_{\mathbf{a},1}$. In particular, if $\mathbf{a} \in N_1$, then no expansion is undertaken in this step. Then let $M^{2n+2}_{\mathbf{a}} \prec M$ be of size χ such that it contains

$$\bigcup_{\mathbf{b} \subsetneq \mathbf{a}} M^{2n+2}_{\mathbf{b}} \cup \bigcup \{M_{i,m} : i < i^*_1, m \in M^{2n+1}_{\mathbf{a}} \text{ for which this is defined}\}$$

Note that the fact we can choose $M^{2n+2}_{\mathbf{a}} \in \mathcal{K}_\chi$ uses that $|i^*_1| \leq \chi$.

At **stage** ω, set $M_{\mathbf{a}} = \cup_{n<\omega} M^n_{\mathbf{a}}$. Note that $\{M_{\mathbf{a}} : \mathbf{a} \in M\}$ forms a cover of M because covers are closed under increasing unions. The first two conditions are satisfied because they were satisfied at stage 0 and no later stage changed $M^0_{\mathbf{a}}$ when $\mathbf{a} \in N_1$. For (3), notice that

$$M_{\mathbf{a}} \cap N_1 = \bigcup_{n<\omega} M^{2n+1}_{\mathbf{a}} \cap N_1 = \bigcup_{n<\omega} M^{2n+1}_{\mathbf{a},1}$$

which is an increasing union of strong substructures of N_1. For (4), let $m \in M_{\mathbf{a}}$ for some $\mathbf{a}$. Then m appears in some $M^{2n+1}_{\mathbf{a}}$. By construction, m does not realize $p \upharpoonright M^{2n+2}_{\mathbf{a}}$. This carries upwards, so m does not realize $p \upharpoonright M_{\mathbf{a}}$.

Now that we have this cover, we can expand M to a τ^+ structure M^+, where F^i_n is n-ary by letting $\{F^i_{\ell(\mathbf{a})} : i < \chi\}$ enumerate $M_{\mathbf{a}}$ such that the first n many functions are projections. The expansions of $M_{\mathbf{a}}$ and $M_{\mathbf{a},1}$ to τ^+ are denoted $M^*_{\mathbf{a}}$ and $M^*_{\mathbf{a},1}$, respectively.

Now, for each $\alpha < (2^\chi)^+$, set M^+_α to be this expansion of M_α. Furthermore, we will denote the parts of the cover as $M_{\alpha,\mathbf{a}}$ and $M_{\alpha,\mathbf{a},1}$ (so their expansion are $M^*_{\alpha,\mathbf{a}}$ and $M^*_{\alpha,\mathbf{a},1}$). Since they never get changed, we require the the expansions of N_0 and N_1 (denoted N^+_0 and N^+_1) are the same in each M^+_α. $\dagger_{\text{Stage 1}}$

Given a τ^+-structure M^+ and $X \subset M^+$, $\mathrm{cl}^{\tau^+}_{M^+}(X)$ denotes the closure of X under the functions of τ^+. By construction, we will have $\mathrm{cl}^{\tau^+}_{M^+}(X) \upharpoonright \tau \prec M^+$.

Stage 2: We want to define some indiscernibles via Morley's Method. Rather than mucking about with nonstandard models of set theory, we use (in a sense) a tree of indiscernibles from M^+ (if that doesn't make sense, ignore it). Recall $\|N_1\| = \lambda$. The goal is to build, for $n < \omega$ and $\alpha < (2^\chi)^+$, injective functions f^n_α with domain $\beth_\alpha(\lambda)$ and range $M_{\beta_n(\alpha)}$ for some $\alpha \leq \beta_n(\alpha) < (2^\chi)^+$ such that

(1) for fixed $\alpha < (2^\chi)^+$ and $n < \omega$, we have that for all $i_1 < \cdots < i_n < \beth_\alpha(\lambda)$, setting $\mathbf{a} = f^n_\alpha(i_1), \dots, f^n_\alpha(i_n)$
  (a) $N^*_{(\alpha,n)} := M^*_{\beta_n(\alpha),\mathbf{a},1}$ is a constant τ^+-substructure of $M_{\beta_n(\alpha)}$; and
  (b) $q^\alpha_n := tp^{\tau^+}_{qf}(\mathbf{a}/N^*_{(\alpha,n)}; M^+_{\beta_n(\alpha)})$ is constant;

(2) for each $n < \omega$, there is some $N^*_{(\cdot,n)} \subset N^+_1$ such that
  (a) $N^*_{(\cdot,0)} \upharpoonright \tau = N_0$;
  (b) for $m < n$, there is a τ^+-embedding $h_{m,n} : N^*_{(\cdot,m)} \to N^*_{(\cdot,n)}$ that form a coherent system;
  (c) for each $\alpha < (2^\chi)^+$, there is $g^n_\alpha : N^*_{(\cdot,n)} \cong N^*_{(\alpha,n)}$; and
  (d) for all $\alpha < (2^\chi)^+$ and $m < n$, there is $\alpha < \beta < (2^\chi)^+$ such that $\langle f^n_\alpha(i) : i < \beth_\alpha(\lambda)\rangle$ is an increasing[14] subset of $\langle f^m_\beta(i) : i < \beth_\beta(\lambda)\rangle$ and the following commutes

$$\begin{array}{ccc} N^*_{(\cdot,m)} & \xrightarrow{h_{m,n}} & N^*_{(\cdot,n)} \\ \downarrow{\scriptstyle g^m_\beta} & & \downarrow{\scriptstyle g^n_\alpha} \\ N^*_{(\beta,m)} & \xrightarrow{id} & N^*_{(\alpha,n)} \end{array}$$

  ; and

(3) fixing $n < \omega$, for each $\alpha < (2^\chi)^+$, we have that

$$q_n := (g^n_\alpha)^{-1}(q^\alpha_n) \in S(N^*_{(\cdot,n)})$$

is constant (as a syntactic type), as is

$$p^1_{(i,n)} := (g^n_\alpha)^{-1}\left(p^1_i \upharpoonright (N^*_{(\alpha,n)} \upharpoonright \tau)\right) \in \mathrm{gS}(N^*_{(\cdot,n)} \upharpoonright \tau)$$

for each $i < i^1_*$ (as a Galois type in $\mathcal{K}$).

The construction of this is standard; one thing to note is the fixing of the Galois type in (3). In Stage 3, the syntactic types will correspond to Φ and the Galois types will correspond to pieces of p^1_i.

Construction: We do this by induction on $n < \omega$ and, inside that, on $\alpha < (2^\chi)^+$.

$\mathbf{n = 0}$: For this case, there's not much to do: $N^*_{(\alpha,0)}$ always has universe N_0 and we can pick g^0_α to be the identity. Set $\beta_0(\alpha) = \alpha$ and let f^0_α enumerate M_α.

[14]According to the order inherited by the enumerations

$\mathbf{n+1}$: This is where it gets fun and, more importantly, where we see the importance of our cardinal arithmetic.

Fix $\alpha < (2^\chi)^+$. First, we color $n+1$-tuples from $\{f^n_{\alpha+\omega}(i) : i < \beth_{\alpha+\omega}(i)\}$ with their qf-type over N_1^+; recall that the n-tuples all have the same type by construction. Erdős-Rado tells us that $\beth_{\alpha+\omega}(\lambda) \to (\beth_\alpha(\lambda))^{n+1}_{2^\lambda}$; well, really it says $\beth_{\alpha+n}(\lambda)^+ \to (\beth_\alpha(\lambda)^+)^{n+1}_{\beth_\alpha(\lambda)}$, but this follows. Thus, we can find $Y^{n+1}_\alpha \subset \beth_{\alpha+\omega}(\lambda)$ such that this type is constant. Note that this already gives us (1) one the construction: $\{f^n_{\alpha+\omega}(i) : i \in Y^{n+1}_\alpha\}$ are $n+1$-indiscernibles over N_1^+, so each $M^*_{\beta_n(\alpha),\mathbf{a},1}$ and $tp^{\tau^+}_{qf}(\mathbf{a}/N^*_{(\alpha,n)}; M^+_{\beta_n(\alpha)})$ are constant for all $n+1$-tuples $\mathbf{a}$ that are increasing from $\{f^n_{\alpha+\omega}(i) : i \in Y^{n+1}_\alpha\}$. Call these $\hat{N}^*_{(\alpha,n+1)}$ and $\hat{q}^\alpha_{n+1}$ for now; not every α will make it and there's some reindexing, so it's premature to define the unhatted version yet.

From this, we have that $\hat{N}^*_{(\alpha,n+1)} \supset N^*_{(\alpha+\omega,n)}$. Now, color each $\alpha < (2^\chi)^+$ with the isomorphism type of $\hat{N}^*_{(\alpha,n+1)}$ over $N^*_{(\alpha+\omega,n)}$ through $(g^n_{\alpha+\omega})^{-1}$; this needlessly obtuse phrase means that we extend $(g^n_{\alpha+\omega})^{-1}$ to an isomorphism containing the $\hat{N}^*_{(\alpha,n+1)}$ in the domain (call this t_α in a notational respite) and we compare isomorphism types of

$$\left\{\left(t_\alpha(\hat{N}^*_{(\alpha,n+1)}), N^*_{(\cdot,n)}\right) : \alpha < (2^\chi)^+\right\}$$

We color $(2^\chi)^+$ many things with $\leq 2^\chi$ many colors, so we can find $X^0_{n+1} \subset (2^\chi)^+$ of size $(2^\chi)^+$ such that this isomorphism type is constant. Once we've fixed this set, we can fix a representative of this class $N^*_{(\cdot,n+1)}$ (for instance, $\hat{N}^*_{(\min X^0_{n+1},n+1)}$); a τ^+-embedding $h_{n,n+1} := g^n_{\min X^0_{n+1}}$ from $N^*_{(\cdot,n)}$ to $N^*_{(\cdot,n+1)}$–from which we form the rest of the $h_{m,n+1}$–; and isomorphisms $\hat{g}^{n+1}_\alpha : N^*_{(\cdot,n+1)} \cong \hat{N}^*_{(\alpha,n+1)}$ such that the following picture commutes

$$\begin{tikzcd}
N^*_{(\cdot,n)} \arrow[r, "h_{n,n+1}"] \arrow[d, "g^n_{\alpha+\omega}"'] & N^*_{(\cdot,n+1)} \arrow[d, "\hat{g}^{n+1}_\alpha"'] \\
N^*_{(\alpha+\omega,n)} \arrow[r] & \hat{N}^*_{(\alpha,n+1)}
\end{tikzcd}$$

To find $\hat{g}^{n+1}_\alpha$, use the fact that $\alpha, \min X^0_{n+1} \in X^0_{n+1}$ to find

$$s_\alpha : t_\alpha(\hat{N}^*_{(\alpha,n+1)}) \cong_{N_{(\cdot,n)}} t_{\min X^0_{n+1}}(N^*_{(\cdot,n+1)})$$

Then set $\hat{g}^{n+1}_\alpha := t_\alpha^{-1} \circ s_\alpha^{-1} \circ t_{\min X^0_{n+1}}$ and chase the following diagram

$$\begin{tikzcd}
\hat{N}^*_{(\alpha,n+1)} \arrow[r, "t_\alpha"] & t_\alpha(\hat{N}^*_{(\alpha,n+1)}) \arrow[rr, "s_\alpha"] & & t_{\min X^0_{n+1}}(N^*_{(\cdot,n+1)}) & N^*_{(\cdot,n+1)} \arrow[l, "t_{\min X^0_{n+1}}"'] \\
N^*_{(\alpha+\omega,n)} \arrow[u] \arrow[rr, "(g^n_{\alpha+\omega})^{-1}"] & & N_{(\cdot,n)} \arrow[ul] \arrow[ur] & & N^*_{(\min X^0_{n+1}+\omega,n)} \arrow[u] \arrow[ll, "(g^n_{\min X^0_{n+1}+\omega})^{-1}"']
\end{tikzcd}$$

This guarantees (2). We shrink again to get (3), but this part will give us (2) in any set we shrink to.

Now color each $\alpha \in X^0_{n+1}$ with the pair

- $(\hat{g}^{n+1}_\alpha)^{-1}(\hat{q}^\alpha_{n+1})$; and

- $(\hat{g}^n_\alpha)^{-1}\left(p^1_i \restriction (\hat{N}^*_{(\alpha,n+1)} \restriction \tau)\right)$

Again, there are $(2^\chi)^+$ many objects colored with 2^χ many colors, so there is $X^1_{n+1} \subset X^0_{n+1}$ such that each of these are constant.

Now we are ready to pick our final sets. We have sets that Y^{n+1}_α of order type $\beth_\alpha(\lambda)$ and X^1_{n+1} of order type $(2^\chi)^+$. For some j in the proper set, we will use $Y^{n+1}_\alpha(j)$ and $X^1_{n+1}(j)$ to denote the jth element of that set under the only possible ordering (the ordering inherited from the ordinals). Thus, we finish by setting, for each $\alpha < (2^\chi)^+$ and $i < \beth_\alpha(\lambda)$,

- $\beta_{n+1}(\alpha) := \beta_n\left(X^1_{n+1}(\alpha) + \omega\right)$
- $f^{n+1}_\alpha(i) := f^n_{X^1_{n+1}(\alpha)+\omega}\left(Y^{n+1}_{X^1_{n+1}(\alpha)}(i)\right)$
- $N^*_{(\alpha,n+1)} := \hat{N}^*_{(X^1_{n+1}(\alpha),n+1)}$
- $q^\alpha_{n+1} = \hat{q}^{X^1_{n+1}(\alpha)}_{n+1}$
- $g^{n+1}_\alpha = \hat{g}^{n+1}_{X^1_{n+1}(\alpha)}$
- $q_{n+1} = (g^{n+1}_\alpha)^{-1}(q^\alpha_{n+1})$
- $p^1_{(i,n+1)}) = (g^{n+1}_\alpha)^{-1}(p^1_i(N^*_{(\alpha,n+1} \restriction \tau))$

noting that the last two items don't depend on α. This is a notational mess, but we essentially just replace every instance of α by the αth member of X^1_{n+1} and every instance of i by the ith member of Y^{n+1}_α.

Then this works. $\dagger_{\text{Construction, Stage 2}}$

Stage 3: Here, we use the objects constructed in Stage 2 to define the appropriate Φ.

First, we want to show that both the q_n's and $p^1_{(i,n)}$'s are increasing with n (after being hit with $h_{m,n}$).

Claim 5.40. *For every $s \subset n$ with $|s| = m$, $q^s_n \restriction \left(h_{m,n}(N^*_{(\cdot,m)})\right) = h_{m,n}(q_m)$. In particular, $h_{m,n}(q_m) \subset q_n$.*

Proof of Claim 5.40: Set $s = \{s_1 < \cdots < s_m\} \subset n$. Fix $\alpha < (2^\chi)^+$ and $i_1 < \cdots < i_n < \beth_\alpha(\lambda)$ and write $\mathbf{a} = f^n_\alpha(i_1), \dots, f^n_\alpha(i_n)$. By (2.b), there is $\beta > \alpha$ and $j_1 < \cdots < j_m < \beth_\beta(\lambda)$ such that $f^n_\beta(j_\ell) = f^n_\alpha(i_{s_\ell})$ for $\ell \leq n$. Then

$$\begin{aligned}
q_m &= (g^m_\beta)^{-1}\left(tp^{\tau^+}_{qf}(f^m_\beta(j_1), \dots, f^m_\beta(j_m)/N^*_{(\beta,m)}, M^+_{\beta_m(\beta)})\right) \\
&= (g^m_\beta)^{-1}\left(tp^{\tau^+}_{qf}(\mathbf{a}^s/N^*_{(\beta,m)}, M^+_{\beta_m(\beta)})\right) \\
&= h^{-1}_{m,n} \circ (g^n_\alpha)^{-1}\left(tp^{\tau^+}_{qf}(\mathbf{a}^s/N^*_{(\alpha,n)}, M^+_{\beta_m(\beta)}) \restriction N^*_{(\beta,m)}\right) \\
&= h^{-1}_{m,n} \circ (g^n_\alpha)^{-1}\left((q^\alpha_n)^s\right) \restriction (g^n_\alpha)^{-1}(N^*_{(\beta,m)}) \\
&= h^{-1}_{m,n}\left(q^s_n \restriction N^*_{(\cdot,m)}\right) \\
h_{m,n}(q_m) &= q^s_n \restriction N^*_{(\cdot,m)}
\end{aligned}$$

as desired. $\dagger_{\text{Claim 5.40}}$

Claim 5.41. *Let $i < i^1_*$. For $m < n$, $p^1_{(i,n)} \restriction \left(h_{m,n}(N^*_{(\cdot,m)}) \restriction \tau\right) = h_{m,n}(p^1_{(i,m)})$.*

Proof of Claim 5.41: This is similar to the above, but without mucking around with the f^n_α's. Let $\alpha < (2^\chi)^+$ and let β be as in (2.b), although we only use the commutative diagram. Then

$$\begin{aligned} p^1_{(i,m)} &= (g^m_\beta)^{-1}\left(p^1_i \upharpoonright (N^*_{(\beta,m)} \upharpoonright \tau)\right) \\ &= h^{-1}_{m,n} \circ (g^n_\alpha)^{-1}\left(\left[p^1_i \upharpoonright (N^*_{(\alpha,n)} \upharpoonright \tau)\right] \upharpoonright (N^*_{(\beta,m)} \upharpoonright \tau)\right) \\ &= h^{-1}_{m,n}((g^n_\alpha)^{-1}\left(p^1_i \upharpoonright (N^*_{(\alpha,n)} \upharpoonright \tau)\right) \upharpoonright (g^n_\alpha)^{-1}(N^*_{(\beta,m)} \upharpoonright \tau)) \\ h_{m,n}(p^1_{(i,m)}) &= p^1_{(i,n)} \upharpoonright (N^*_{(\cdot,m)} \upharpoonright \tau) \end{aligned}$$

$\dagger_{\text{Claim 5.41}}$

This means that the sequences $\{h^{-1}_{0,n}(q_n) : n < \omega\}$ and $\{h^{-1}_{0,n}(N^*_{(\cdot,n)}) : n < \omega\}$ are increasing. Remove this directed nonsense by setting $\bar{q}_n := h^{-1}_{0,n}(q_n)$ and $\bar{N}^*_{(\cdot,n)} := h^{-1}_{0,n}(N^*_{(\cdot,n)})$; note that the first is increasing by Claim 1 and the second is increasing by construction.

Now set $\Phi = \{\bar{q}_n \mid n < \omega\}$ and $\bar{N}^*_{(\cdot,\omega)} = \cup_{n<\omega}\bar{N}^*_{(\cdot,n)}$. Moreover, $\langle \bar{N}^*_{(\cdot,n)} \upharpoonright \tau : n < \omega\rangle$ is a $\prec_\mathcal{K}$-increasing sequence, so $\bar{N}^*_{(\cdot,\omega)} \upharpoonright \tau \in \mathcal{K}$; however, there's no reason to expect that $\bar{N}^*_{(\cdot,\omega)} \upharpoonright \tau \prec M_\alpha$ for any of the α's. Set $\bar{N}_{(\cdot,n)} := \bar{N}^*_{(\cdot,n)} \upharpoonright \tau$ and similarly for $\bar{N}_{(\cdot,\omega)}$. Moreover, $\bar{q}_n$ is a type over $\bar{N}^*_{(\cdot,n)}$, so we can add a constant to the language of Φ for each element of $\bar{N}^*_{(\cdot,n)}$. The following claim says that this changes nothing.

Claim 5.42. *Φ is a template proper for linear orders in $\mathcal{K}$ such that $\tau(\Phi)$ has constants for every element in $\bar{N}_{(\cdot,\omega)}$; one could write this as $\Phi \in \Upsilon_\chi[\mathcal{K}_{\bar{N}_{(\cdot,\omega)}}]$.*

Proof of Claim 5.42: That Φ is a template for $\mathcal{K}$ (rather than $\mathcal{K}_{\bar{N}_{(\cdot,\omega)}}$) already follows. The only potential problem in the additional step is that, for $n < m$, q_n doesn't specify the diagram over $N^*_{(\cdot,m)}$. However, using Claim 1, we can see this is fine because any way of enlarging an n-tuple to an m-tuple gives the same q_m type, which specifies this diagram. $\dagger_{\text{Claim 5.42}}$

Thus, we have that, for any I, $EM_{\tau(N_{(\cdot,\omega)})}(I, \Phi) \in \mathcal{K}_{\bar{N}_{(\cdot,\omega)}}$. This gives a canonical isomorphism of $\bar{N}_{(\cdot,\omega)}$ into $EM_\tau(I, \Phi)$, so we will assume that this is just the identity.

We have now defined everything from the theorem statement: N'_n is (the canonical copy of) $\bar{N}^*_{(\cdot,n)}$ in $EM_\tau(n, \Phi)$ and $p^1_{i,n}$ is (the corresponding copy of) $h^{-1}_{0,n}(p^1_{(i,n)}) \in \mathrm{gS}(\bar{N}^*_{(\cdot,n)} \upharpoonright \tau)$. The first three conditions are clear. The omission of Γ_0 is standard: given $\mathbf{a} \in EM_\tau(I, \Phi)$, we have that $\mathbf{a} \in EM_\tau(J, \Phi)$ for some finite $J \subset I$. Then we can find $f : EM_\tau(J, \Phi) \to_{N_0} M$ by construction. Then, $EM_\tau(J, \Phi)$ omits Γ_0 since M does. Since Γ_0 are types over N_0, this is preserved by f, so $\mathbf{a}$ doesn't realize any type in Γ_0.

The final piece of the theorem is contained in the next claim.

Claim 5.43. *Fix $i < i^1_*$ and let $p^1_{(i,\omega)}$ be any type over N'_ω that extends each $p^1_{(i,n)}$. For any infinite I, $EM_\tau(I, \Phi)$ omits each $p^1_{(i,\omega)}$. In particular, if finite $J \subset I$ and $x \in EM_\tau(J, \Phi)$, then x does not realize $p^1_{(i,|J|)}$.*

Proof of Claim 5.43: Let $J \subset I$ be finite with $n := |J|$. Then

$$J \vDash \bar{q}_n = h^{-1}_{0,n} \circ (g^n_\alpha)^{-1}\left(tp^{\tau^+}_{qf}(\mathbf{a}/N^*_{(\alpha,n)}; M^+_{\beta_n(\alpha)})\right)$$

where $\mathbf{a} = f^n_\alpha(i_1), \ldots, f^n_\alpha(i_n)$ for some/any $\alpha < (2^\chi)+$ and $i_1 < \cdots < i_n < \beth_\alpha(\lambda)$; the some/any doesn't matter because of the construction, especially (1). This equality of quantifier free types (pushed from $(g^n_\alpha)^{-1}$) gives rise to a τ^+-isomorphism

$$h : EM_\tau(J, \Phi) \cong \mathrm{cl}^{\tau^+}_{M^+_{\beta_n(\alpha)}}(\mathbf{a})$$

that extends $h^{-1}_{0,n} \circ (g^n_\alpha)^{-1}$. At long last, reaching back to (4) from the **Stage 1**, we obtain that $\mathrm{cl}^{\tau^+}_{M^+_{\beta_n(\alpha)}}(\mathbf{a}) \restriction \tau$ omits the Galois types $p^1_i \restriction N_{(\alpha,n)}$ for each $i < i^1_*$ (recalling here that $N_{(\alpha,n)} = M^*_{\beta_n(\alpha),\mathbf{a},1}$). Hitting this with h (and recalling that it extends $h^{-1}_{0,n} \circ (g^n_\alpha)^{-1}$), we get that $EM_\tau(J, \Phi)$ omits

$$h^{-1}\left(p^1_i \restriction N_{(\alpha,n)}\right) = h^{-1}_{0,n} \circ (g^n_\alpha)^{-1}\left(p^1_i \restriction N_{(\alpha,n)}\right) = \bar{p}^1_{(i,n)} = p^1_{i,n}$$

as desired. †Claim 5.43, Stage 3, Theorem 5.39

Now we can prove exactly what we need.

Definition 5.44. (1) *We say that μ is a χ-*collection cardinal *iff for all $\kappa < \mu$, $\beth_{(2^\chi)^+} \leq \mu$.*
(2) *If $\mathcal{K}$ is categorical in $\lambda \geq \beth_{(2^{LS(\mathcal{K})})^+}$, then set $\chi(\mathcal{K})$ to be the minimal χ such that $\mathcal{K}$ is weakly $\left(\chi, [\beth_{(2^{LS(\mathcal{K})})^+}, \lambda)\right)$-tame.*

Note that at least one such χ exists in (2) by Theorem 4.25.

Exercise 5.45. *The second Hanf number $\beth_{\left(2^{\beth_{(2^{LS(\mathcal{K})})^+}}\right)^+}$ is the first $(2^{LS(\mathcal{K})})^+$-collection cardinal.*

Theorem 5.46. *Suppose $\mathcal{K}$ is categorical in λ and has amalgamation. If $\lambda > \beth_{(2^{\chi(\mathcal{K})})^+}$, then $\mathcal{K}$ is categorical in any $\chi(\mathcal{K})$-collection cardinal less than λ.*

Proof: Suppose $\mathcal{K}_\mu$ is not categorical where μ is a $\chi(\mathcal{K})$-collection cardinal $< \lambda$. We know that there is a Galois saturated member of $\mathcal{K}_\mu$ (since Galois stability holds below the categoricity cardinal by Theorem 3.37), so let $M \in \mathcal{K}_\mu$ be a non-Galois saturated model. By our approximation Theorem 5.35, we know that $\mathcal{M}$ is χ^+-saturated. Let κ be minimal such that $\mathcal{M}$ is not κ^+-Galois saturated. Then there is $M_0 \prec M$ of size κ and $p \in \mathrm{gS}(M_0)$ such that p is not realized in M. Without loss of generality, since M is κ-Galois saturated, we can extend M_0 to a Galois saturated model. By definition (or really Theorem 4.25), $\mathcal{K}$ is weakly $(\chi(\mathcal{K}), \kappa)$-tame. Since M_0 is Galois saturated, this means that omitting p is equivalent to omitting $p/E_{\chi(\mathcal{K})}$. So M omits p with κ-sized domain with $\|M\| \geq \beth_{(2^{\chi(\mathcal{K})})^+}(\kappa) \leq \|M\|$; this inequality is by the definition of χ-collection cardinal.

Now, an application of Shelah's Omitting Types Theorem 5.39 gives a $\Phi \in \Upsilon_{\mathrm{LS}(\mathcal{K})}$ such that no $EM_\tau(I, \Phi)$ with I infinite is $\chi(\mathcal{K})^+$-Galois saturated. However, $EM_\tau(\lambda, \Phi)$ must be $\chi(\mathcal{K})^+$-Galois saturated by categoricity. Thus, we have a contradiction to the assumption that there is a non-Galois saturated model in $\mathcal{K}_\mu$. So $\mathcal{K}$ is categorical in μ. †

5.4. **Admitting saturated unions.** We introduce an important hypothesis that is a parameterized version of (an equivalent characterization of) superstability in first-order theories.

Definition 5.47.
(1) *$\mathcal{K}^{sat}$ is the collection of Galois saturated models of $\mathcal{K}$. $\mathcal{K}^{sat}_\kappa$ is the collection of Galois saturated models of $\mathcal{K}_\kappa$.*
(2) $\mathcal{K}$ admits λ saturated unions *iff $\mathcal{K}^{sat}_\lambda$ is nonempty and closed under increasing unions of length less than λ^+.*

(3) $M \in \mathcal{K}$ *is* superlimit *iff it is universal, has a proper extension, and for every increasing sequence* $\langle M_i \mid i < \delta < \|M\|^+\rangle$*, if each* $M_i \cong M$*, then* $\cup_{i<\delta} M_i \cong M$*.*

Preserving λ-saturation for longer than λ-length unions is easy.

Exercise 5.48. *Let* $\langle M_i \mid i < \delta\rangle$ *be a increasing sequence of* λ*-saturated models with* $\lambda \leq cf\,\delta$*. Show that* $\cup_{i<\delta} M_i$ *is* λ*-saturated.*

These notions are related in the following way

categoricity in $\lambda \implies$ admission of λ saturated unions $\implies$ there is a superlimit in $\mathcal{K}_\lambda$

See [BV17a, Section 6] and especially [**?**] for more on the various relations between admitting λ-saturated unions, uniqueness of limit models, and other notions of superstability in tame AECs. One reason to care about these intermediate properties is the following:

Suppose that we have $(\mathbb{K}, \prec_{\mathbb{K}})$ that consists acts like an AEC *except* that it only consists of models of a single size λ and is not closed under unions of length longer λ; we call this an *AEC in* λ. Then, following Exercise 3.27, we can form $(\mathbb{K}^{up}, \prec^{up})$ to be the closure of $\mathbb{K}$ under directed colimits; this is an AEC. Now suppose that we had superlimit $M \in \mathcal{K}_\lambda$ and $\mathbb{K} = \mathcal{K}[M] := \{N \in \mathcal{K}_\lambda \mid M \cong N\}$. Then the superlimit definition implies that this is an AEC in λ. Then $\mathcal{K}[M]^{up}$ is a sub-AEC of $\mathcal{K}_{\geq\lambda}$ that is categorical in λ. In this sense, it forms a 'categorical core,' which indicates there should be some structure there.

Our goal is to show that categoricity implies admission of saturated unions below the categoricity cardinal (Theorem 5.55). To do so, we have to show that we have no long splitting chains. In particular, we introduce the notion of universal local character.

Definition 5.49. *Define* $\kappa^*(\mu)$ *to be the minimum* α *such that if there is an increasing chain* $\langle M_i \in \mathcal{K}_\mu \mid i \leq \delta\rangle$ *with*

(1) $\alpha \leq \delta$*;*
(2) M_0 *is Galois saturated;*
(3) M_{i+1} *is universal over* M_i*; and*
(4) $p \in gS(M_\delta)$*,*

then there is some $i_0 < \delta$ *such that* p μ*-Galois splits over* M_{i_0}*.*

Categoricity and amalgamation implies that this is ω, which means that the conclusion holds for any length chains. For more computations of this value, see [SV99, **?**, **?**] (although take note that more than simply this value is required to prove Theorem 5.55; the recognition of EM models remains a key tool). We revisit out old friend, the highly homogeneous ${}^{<\omega}\mu$ from our discussion of brimful models.

Theorem 5.50 ([She99, Lemma 6.3], [Bal09, Theorem 15.3])**.** *Suppose that* $\mathcal{K}$ *has amalgamation,* $\Phi \in \Upsilon_{LS(\mathcal{K})}[\mathcal{K}]$*, and* $\mu \in (LS(\mathcal{K}), cf\,\lambda)$*. Then* $\kappa^*(\mu) = \omega$*.*

A very important note is that we've introduced the $\mu < \mathrm{cf}\,\lambda$ hypothesis. Previously, we had avoided this with Fact 4.32. However, the proof of Fact 4.32 uses Theorem 5.50. Thus, we must avoid that result and use the coarser Exercise 4.31.

Also, below we use the ordinal $\delta + \mu$ several times. If $\delta < \mu$, then this is just μ.

Proof: Suppose we have a universal chain $\langle M_i \in \mathcal{K}_\mu \mid i \leq \delta\rangle$ and $p \in \mathrm{gS}(M_\delta)$ as in the set-up of Definition 5.49. Assume for sake of contradiction that p μ-Galois splits over every M_i. By Corollary 5.30, the sequence $\langle EM_\tau(\beta \times {}^{<\omega}\mu, \Phi) \mid \beta \leq \delta\rangle$ is also a universal chain. Since the basis are both Galois saturated, Exercise 5.28 tells us that the two different tops are isomorphic. Thus, it is enough to assume that $M_i = EM_\tau(i \times {}^{<\omega}\mu, \Phi)$ (if $\delta = \mathrm{cf}\,\delta$ uncountable, this is an easy club argument; when $\mathrm{cf}\,\delta = \omega$, this requires a little more care).

Now that we are in an EM situation, we can extend this to a universal sequence $M_i = EM_\tau(i \times {}^{<\omega}\mu, \Phi)$ for $i \leq \delta + \mu$. We know that all limit models are Galois saturated by Theorem

5.31. Thus, we can apply Corollary 5.26 to get a nonalgebraic extension $q \in \mathrm{gS}(M_{\delta+\mu})$ of p that does not μ-Galois split over M_β.

Then $EM_\tau(\lambda, \Phi)$ is cf λ-Galois saturated, so we can find a realization of q in $EM_\tau\left((\delta+\mu+n) \times {}^{<\omega}\mu, \Phi\right)$. This is realized by some $\sigma(\mathbf{a}, \mathbf{b})$ for some $\sigma \in \tau(\Phi)$; $\mathbf{a} \in \delta \times {}^{<\omega}\mu$; and $\mathbf{b} \in [\delta, \delta+\mu+n) \times {}^{<\omega}\mu$. We'd now like to extend this EM model to get enough room to move things around. Define

$$f : (\delta+\mu+n) \times {}^{<\omega}\mu \to (\delta+\mu+\delta+\mu+n) \times {}^{<\omega}\mu$$

by fixing $\delta \times {}^{<\omega}\mu$ and sending

$$(\delta+\alpha, s) \in [\delta, \delta+\mu+n) \times {}^{<\omega}\mu \text{ to } (\delta+\mu+\delta+\alpha, s) \in [\delta+\mu, \delta+\mu+\delta+\mu+n) \times {}^{<\omega}\mu$$

Set

$$q' = \mathrm{gtp}\left(\sigma\left(\mathbf{a}, f(\mathbf{b})\right)/M_{\delta+\mu}; EM_\tau\left((\delta+\mu+\delta+\mu+n) \times {}^{<\omega}\mu, \Phi\right)\right)$$

Note that $q' \upharpoonright M_\delta = q \upharpoonright M_\delta$ because everything is of the same order-type. Now we have our final claim.

Claim: q' does not μ-Galois split over M_i for some $i < \delta$.

First observe that this claim will finish the theorem since we have built the Galois types so $q' \upharpoonright M_\delta = q \upharpoonright M_\delta = p$.

To see the proof, fix the minimal $\beta_0 < \delta$ such that $\mathbf{a} \in \beta_0 \times {}^{<\omega}\mu$. For contradiction, suppose that q' does μ-Galois split over M_{β_0}. We will show the following subclaim.

Subclaim: For every $\beta_0 < \alpha < \delta+\mu$, $q' \upharpoonright EM_\tau\left((\alpha+1) \times {}^{<\omega}\mu, \Phi\right)$ does μ-Galois split over $EM_\tau(\alpha \times {}^{<\omega}\mu, \Phi)$.

For the subclaim, fix such an α. To witness the Galois splitting of q' over M_{β_0}, there is $X \subset [\beta+0, \delta+\mu)$ of size μ such that the witnesses appear in $EM_\tau\left((\beta \cup X) \times {}^{<\omega}\mu, \Phi\right)$. Now define

$$g : (\delta+\mu+\delta+\mu+n) \times {}^{<\omega}\mu \to (\delta+\mu+\delta+\mu+n) \times {}^{<\omega}\mu$$

such that

(1) $\beta_0 \times {}^{<\omega}\mu$ is fixed;
(2) $X \times {}^{<\omega}\mu$ is mapped into $\{\alpha\} \times {}^{<\omega}\mu$; and
(3) $[\delta+\mu, \delta+\mu+\delta+\mu+n) \times {}^{<\omega}\mu$ is fixed.

This is straightforward from Proposition 5.15.(2).

Then this ensures the subclaim. $\dagger_{\text{Subclaim}}$

From the Subclaim, we have built a long μ-Galois splitting chain as in Lemma 5.4. But this contradicts μ-Galois stability, which follows from categoricity by Theorem 3.37. $\dagger$

Now we are going to use this to show that categoricity implies union of saturated is saturated. The argument we give follows our tradition of following [Bal09]'s exposition of [She99] (although reorganized a bit). The arguments given there seem to miss a step, but are essentially correct. Since that work, more straightforward proofs that *do not* use EM models have been given. For instance, both [Vasa, Theorem 3.3 and Remark 3.4] and [Van16, Theorem 22] give more direct arguments. Additionally, the methods of [BV17a] could probably be adopted, but that would take much more work. I'd like to thank Sebastien Vasey for help with the fix and pointing out the alternative arguments.

The main feature of our proof will be to build an order property from a failure of admitting saturated unions. In first-order, this immediately implies instability and many models, both of which would be contradictions. In AECs, things are a little trickier because we lack compactness. However, we're able to get around this by building this witness in an EM model and, moreover, by the same terms. This allows us much better portability of notions. To this end, we define the following notion.

Definition 5.51. *For $\mathcal{K}$ with arbitrarily large models, we say that $\mathcal{K}$ has* the blueprint-witnessed (α_1, α_2)-ary order property of size χ *iff there is $\Phi \in \Upsilon[\mathcal{K}]$; linear order I of size $\leq \chi$; and α_ℓ-tuples of terms $\bar{\sigma}_\ell(\mathbf{x}) \in \tau(\Phi)$ of the same arity such that, for any J that is an end extension of I and $\mathbf{i}_1 < \mathbf{i}_2, \mathbf{j}_1 < \mathbf{j}_2 \in J - I$, we have that*

$$gtp\left(\bar{\sigma_1}(\mathbf{i}_1)\bar{\sigma_2}(\mathbf{i}_2)/EM_\tau(I, \Phi); EM_\tau(J, \Phi)\right) \neq gtp\left(\bar{\sigma_1}(\mathbf{j}_2)\bar{\sigma_2}(\mathbf{j}_1)/EM_\tau(I, \Phi); EM_\tau(J, \Phi)\right)$$

The blueprint witness gives us the same ability to port these witnesses around that we have with compactness. For instance, the following is basically the same as the first-order proof.

Theorem 5.52. *Suppose that $\mathcal{K}$ has the blueprint-witnessd $(1, \leq \chi)$-ary order property of size χ; write σ_1 for $\bar{\sigma}_1$ to denote that it's a single element. Then for κ, there is $M \in \mathcal{K}_{2^{<\kappa}+\chi}$ such that $|gS(M)| \geq 2^\kappa$.*

Proof: Let $\Phi \in \Upsilon[\mathcal{K}]$, I, and $\bar{\sigma}_\ell$ witness and let J_0 be the arity of the $\bar{\sigma}_\ell$ (written in increasing order). Then consider the linear orders

$$I^\frown \left({}^{<\kappa}2 \times J_0\right) \subset I^\frown \left({}^{\leq\kappa}2 \times J_0\right)$$

For $\eta \in {}^\kappa 2$, set $a_\eta = \sigma_1\left(\{\eta\} \times J_0\right)$, the η version of the singleton witnessing the order property. We will show that

$$\left\{\mathrm{gtp}\left(a_\eta/EM_\tau\left(I^\frown\left({}^{<\kappa}2 \times J_0\right), \Phi\right)\right) \mid \eta \in {}^\kappa 2\right\}$$

are pairwise distinct. Let $\eta \neq \nu \in {}^\kappa 2$ and set $\rho = \eta \cap \nu$. WLOG, $\rho^\frown\langle 1\rangle < \nu$. Set $X_\rho := \bar{\sigma}_2\left(\{\rho^\frown\langle 1\rangle\} \times J_0\right)$. Note that, by construction,

$$\{\eta\} \times J_0 < \{\rho\} \times J_0 < \{\nu\} \times J_0$$

Then by the definition of the order property, we have that

$$\mathrm{gtp}\left(a_\eta X_\rho/EM_\tau(I, \Phi); EM_\tau\left(I^\frown\left({}^{\leq\kappa}2 \times J_0\right), \Phi\right)\right) \neq \mathrm{gtp}\left(a_\nu X_\rho/EM_\tau(I, \Phi); EM_\tau\left(I^\frown\left({}^{\leq\kappa}2 \times J_0\right), \Phi\right)\right)$$

Since each $X_\rho \subset EM_\tau\left(I^\frown\left({}^{<\kappa}2 \times J_0\right), \Phi\right)$, we have shown all of these types are distinct and contradicted stability. †

Shelah claims even better, that the order property implies many models [She99, Claim 4.8]. We can ask how we get a blueprint-witnessed order property. The standard approach is to get a Hanf number length normal order property and then use Morley's Omitting Types Theorem 3.33. However, in the following, we will use our affinity for recognizing EM models to do it by hand.

We first prove that $\mathcal{K}$ admits unions of saturated models at regular cardinals. This regularity seems to be necessary for the arguments of Baldwin and Shelah. However, as we will show in Theorem 5.55, this will be enough.

Lemma 5.53. *Suppose that $\mathcal{K}$ has amalgamation, is λ-categorical, has arbitrarily large models. If $\mu \in (LS(\mathcal{K}), cf\ \lambda)$ is regular, then then $\mathcal{K}$ admits μ saturated unions.*

Proof: Let $\langle N_i \in \mathcal{K}^{sat}_\mu \mid i < \delta\rangle$ be increasing for $\delta = \mathrm{cf}\ \delta < \mu^+$, and set $N_\delta = \cup_{i<\delta} N_i$. For sake of contradiction, suppose this is not Galois saturated. Then there is $M^- \prec N$ of size $\chi < \mu$

and $p \in \mathrm{gS}(M^-)$ not realized in N. By Exercise 5.48, we may further assume $\delta < \chi$. Since p is not realized in N, there must be some nonalgebraic extensions

$$\hat{p} = \mathrm{gtp}(d/N; N^*) \in \mathrm{gS}^{na}(N)$$

Step 1 is to construct a sequence $\langle M_i \in \mathcal{K}_\chi^{sat} \mid i \leq \delta\rangle$ and $\{M_i^+ \in \mathcal{K}_\chi \mid i < \delta\}$ such that

(1) $M^- \prec M_\delta$ and $M_i \prec N_i$;
(2) M_{i+1} is universal over M_i;
(3) if $\hat{p}$ χ-Galois splits over M_i, then $\hat{p} \upharpoonright M_i^+$ χ-Galois splits over M_i; and
(4) if $j < i < \delta$, then $N_i \cap M_j^+ \subset M_{i+1}$

This is straightforward remembering that each N_i is Galois saturated and that we have proved that χ-limit models are unique (Theorem 5.31). By Theorem 5.50, there must be some $i_0 < \delta$ such that $\hat{p} \upharpoonright M_\delta$ does not χ-Galois split over M_{i_0}. By construction, $M_{i_0}^+ \prec M_\delta$, so this implies that $\hat{p}$ does not χ-Galois split over M_{i_0}.

Step 2 (the longer step) is to build the blueprint-witnessed order property in this set-up. By the appropriate generalization of Corollary 4.34 to our situation (where we have cf λ-Galois saturation only) and the argument in the first paragraph in the proof of Theorem 5.25, we have that

$$(N_{i_0}, M_{i_0}) \cong (EM_\tau(\mu, \Phi), EM_\tau(\chi, \Phi))$$

We assume this isomorphism is the identity. Then $N_{i_0} \prec EM_\tau(\lambda, \Phi)$, so there is

$$h : N^* \to_{M_{i_0}} EM_\tau(\lambda, \Phi)$$

Then there are $X_0 \subset [\chi, \mu)$ and $X_1 \subset [\mu, \lambda)$ of size χ such that

$$h(M_\delta) \cup \{h(d)\} \subset EM_\tau(\chi \cup X_0 \cup X_1, \Phi)$$

Since $\chi < \mu = \mathrm{cf}\,\mu$, there is some $\beta_0 < \mu$ such that $X_0 \subset \beta_0$; this is the crucial use of the regularity of μ. We can similarly arrange the $EM_\tau(\beta, \Phi)$ is universal over $EM_\tau(\chi, \Phi)$.

Write

- $\gamma := \mathrm{otp}(X_1)$;
- $h(d) = \sigma_1(\mathbf{a}, \mathbf{b})$ for $\mathbf{a} \in \beta_0$ and $\mathbf{b} \in X_1$; and
- $h(M_\delta) = \bar{\sigma}_2(\mathbf{a}', \mathbf{b}')$ for $\mathbf{a}' \in [\beta_0]^{\leq\chi}$ and $\mathbf{b}' \in [X_1]^{\leq\chi}$.

Now, we define the following:

(1) $\gamma_\alpha = \alpha \cdot \gamma$;
(2) $g_\alpha : \beta_0 + \gamma_\alpha \cup X_1 \mapsto_{\beta_0+\gamma_\alpha} \beta_0 + \gamma_{\alpha+1}$ is order-preserving;
(3) $\hat{g}_\alpha$ is the lifting of g_α to a map

$$\hat{g}_\alpha : EM_\tau(\beta_0 + \gamma_\alpha \cup X_1, \Phi) \cong_{EM_\tau(\beta_0+\gamma_\alpha,\Phi)} EM_\tau(\beta_0 + \gamma_{\alpha+1}, \Phi)$$

(4) $M^\alpha := \hat{g}_\alpha \circ h(M_\delta) \prec EM_\tau(\beta_0 + \gamma_{\alpha+1}, \Phi)$; and
(5) $a^\alpha := \hat{g}_\alpha \circ h(d) \in EM_\tau(\beta_0 + \gamma_{\alpha+1}, \Phi)$.

We wish to show that these elements exhibit the order property, e. g., that we can tell the order of α and β by examining $\mathrm{gtp}(a^\alpha M^\beta/M_{i_0}; N_{i_0}$.

- <u>If $\alpha < \beta$,</u> both h and $\hat{g}_\beta$ fix $a^\alpha \in N_{i_0}$. Since $p = \mathrm{gtp}(d/M_\delta; N^*)$, we have

  $$\mathrm{gtp}(dM_\delta/M_{i_0}; N_*) \neq \mathrm{gtp}(a^\alpha M_\delta/M_{i_0}; N) = \mathrm{gtp}(a^\alpha M^\beta/M_{i_0}; N_{i_0})$$

  where we have hit the middle equation with M_{i_0}-fixing $\hat{g}_\beta \circ h$ to get the right equality.
- <u>If $\beta < \alpha$,</u> then we have built things so a^α realizes $\mathrm{gtp}(d/M^\beta; N^*)$. Thus,

  $$\mathrm{gtp}(a^\alpha M^\beta/M_{i_0}; N_{i_0}) = \mathrm{gtp}(dM^\beta/M_{i_0}; N^*)$$

Since $\text{gtp}(d/N; N^*)$ does not χ-Galois split over M_{i_0}, we have that

$$\text{gtp}(dM_\delta/M_{i_0}; N^*) = \text{gtp}(dM^\beta/M_{i_0}; N^*)$$

Thus, by transitivity of equality,

$$\text{gtp}(dM_\delta/M_{i_0}; N^*) = \text{gtp}(a^\alpha M^\beta/M_{i_0}; N_{i_0})$$

Putting this together, for any $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, we have that

$$\text{gtp}\left(a^{\alpha_1} M^{\beta_1}/M_{i_0}; N_{i_0}\right) \neq \text{gtp}\left(dM_\delta/M_{i_0}; N^*\right) = \text{gtp}\left(a^{\beta_2} M^{\alpha_2}/M_{i_0}; N_{i_0}\right)$$

Since this was all built in accordance with a blueprint, we have the $(1, \chi)$-ary order property of size χ. By Theorem 5.52, this gives Galois instability in many cardinals, for instance χ. But by categoricity and Theorem 3.37, $\mathcal{K}$ is χ-Galois stable, a contradiction. †

Now we get rid of the need to use regularity. First, a lemma that the only thing that matters is the level of saturation, not the size of the model.

Lemma 5.54. *Suppose that $\mathcal{K}$ is an AEC with amalgamation that admits μ saturated unions and is μ-Galois stable. Then $\mathcal{K}^{\mu-sat}_{\geq\mu}$ is closed under unions of chains.*

Proof: Let $\langle M_i \in \mathcal{K}^{\mu-sat}_{\geq\mu} \mid i < \alpha = \text{cf } \alpha\rangle$ be $\prec$-increasing. Then $M_\alpha := \cup_{i<\alpha} M_i \in \mathcal{K}$ by the axioms of an AEC. If $\alpha > \mu$, then M_α is μ-Galois saturated by cofinality arguments. So suppose that $\alpha \leq \mu$ and let $N \prec M_\alpha$ of size $< \mu$. Then, we can use stability to find $\prec$-increasing $\langle N_i \in \mathcal{K}^{sat}_\mu \mid i < \alpha\rangle$ such that

(1) $N_i \prec M_i$; and
(2) $N \cap M_i \subset N_i$.

Since $\mathcal{K}$ admits μ saturated unions, $N_\alpha := \cup_{i<\alpha} N_i$ is Galois saturated. In particular, it realizes every type over N. Since $N_\alpha \prec M_\alpha$, so does M_α. Thus M_α is μ-Galois saturated. †

Theorem 5.55. *Suppose that $\mathcal{K}$ has amalgamation, is λ-categorical, has arbitrarily large models. If $\mu \in (LS(\mathcal{K}), cf\ \lambda)$, then then $\mathcal{K}$ admits μ saturated unions.*

Proof: We have the result for all regular μ by Lemma 5.53. If μ is singular, then it is limit and μ-Galois saturation is equivalent to being $(\kappa + \text{LS}(\mathcal{K}))^+$-Galois saturated for all $\kappa < \mu$. Then this is enough by Lemma 5.54. †

5.5. **Vaughtian pairs.** The final ingredient is the notion of a Vaughtian pair. Remember in first-order, a Vaughtian pair is $M \precneqq N$ along with a formula ϕ such that $\phi(M) = \phi(N)$ is infinite. We might tag this with data $(\phi, \|M\|)$ to make it more explicit (note N can always be chosen to have the same size as M). Then, Baldwan and Lachlan [BL71] famously characterized uncountably categorical countable first-order theories as those that are ω-stable and have no Vaughtian pairs. This can be generalized to AECs as follows.

Definition 5.56.

(1) *$M \precneqq N$ is a* (p, λ)-Vaughtian pair *iff*
   (a) $\|M\| = \|N\| = \lambda$;
   (b) *dom* $p \prec M$;
   (c) *p has a nonalgebraic extension to M; and*
   (d) $p(M) = p(N)$.
(2) *$M \prec N$ is a* true (p, λ)-Vaughtian pair *iff it is a Vaughtian pair with both models Galois saturated.*

There are two main parts to the argument with Vaughtian pairs. First, we will show that categoricity in a successor will imply no true Vaughtian pairs in the predecessor. Then we will prove a great deal of transfer results related to changing the cardinalities of the models in the Vaughtian pairs. This will culminate in Lemma 5.65 that will allow us to transfer categoricity.

Proposition 5.57. *Suppose $\mathcal{K}$ is an AEC with amalgamation that is categorical in $\lambda^+ > LS(\mathcal{K})^+$ and is weakly (χ, λ)-tame for $\chi < \lambda$. If $M \in \mathcal{K}^{sat}_\lambda$ and $p \in gS(M)$ is minimal, then there is no true (p, λ)-Vaughtian pair.*

Proof: Suppose that $N_0 \prec N_1$ are a true (p, λ)-Vaughtian pair. By Theorem 5.55, $\mathcal{K}$ admits λ saturated unions. Let $p_0 \in \mathrm{gS}^{na}(N_0)$ extend p. Since $p(N_1) = p(N_0)$, p_0 is not realized in N_1. Now find $N \prec N' \prec N_0$ such that

(1) p_0 does not LS($\mathcal{K}$)-Galois split over N;
(2) N' is universal over N; and
(3) $\|N'\| < \|N_0\|$.

We can find N by stability using Existence (Theorem 5.5) and N' using the Galois saturateion of M. Now we build an increasing, continuous chain $\langle N_i \in \mathcal{K}^{sat}_\lambda \mid i < \lambda^+\rangle$ and isomorphisms $f_i : (N_1, N_0) \cong_{N'} (N_{i+1}, N_i)$. This is possible using the uniqueness of saturated models and then crucially the admission of λ saturated unions to move through limit stages.

Set $p_i = f_i(p_0)$. Note that p_i is not realized in N_{i+1} because of the isomorphism f_i. Also, $p_i \upharpoonright N' = p_0 \upharpoonright N'$, so by Uniqueness (Theorem **??**), $p_i \upharpoonright N_0 = p_0$. By the minimality of p and **??**, p_i is the unique nonalgebraic extension of p_0.

Set $N_{\lambda^+} := \bigcup_{i<\lambda^+} N_i \in \mathcal{K}_{\lambda^+}$. By λ^+-categoricity, N_{λ^+} is Galois saturated, so there is $a \in N_{\lambda^+}$ realizing p_0. But $a \in N_{i+1} - N_i$ for some $i < \lambda^+$. Then a realizes p_i, a contradiction. †

We will deal extensively with minimal types. Recall from Definition 5.19 that minimal types have exactly one big extension to any larger model. This talk of big types complicates the matter, but Lemma 5.32 greatly simplifies things by showing big types to be the same as nonalgebraic under certain hypotheses. We can easily derive the following:

Corollary 5.58. *Suppose $\mathcal{K}$ is an AEC with amalgamation that is χ-Galois stable and weakly (χ, μ)-tame. If $M \in \mathcal{K}^{sat}_\mu$ and minimal $p \in gS^{na}(M)$ with $M \prec N \in \mathcal{K}^{sat}_\mu$, then p has a unique nonalgebraic extension to N, which is minimal.*

Proposition 5.59. *Suppose $\mathcal{K}$ is an AEC with amalgamation that is λ-categorical and weakly $(\chi, [\mu, \lambda])$-tame with $\chi < \mu \le \kappa \le \lambda$. Let $M \prec N$ from $\mathcal{K}_\kappa$ and minimal $p \in gS(M)$ with an extenstion $q \in gS(N)$ that does not μ-Galois split over M. If there is a (q, λ)-Vaughtian pair, then there is a (p, λ)-Vaughtian pair.*

Proof: Without loss of generality, assume that N is Galois saturated, and let $N_0 \prec N_1$ by a (q, λ)-Vaughtian pair. If they are not also a (p, λ)-Vaughtian pair, then there is $b \in p(N_1) - p(N_0)$. Then b cannot realize q. Thus, $\mathrm{gtp}(b/N; N_1)$ and q are distinct nonalgebraic extensions of p. By weak tameness, this contradicts the minimality of p. †

Proposition 5.60. *Suppose that $\mathcal{K}$ is an AEC. If there is $M \in \mathcal{K}_\lambda$ with big $p \in gS(M)$ that has no (p, λ)-Vaughtian pairs, then any $M \prec N \in \mathcal{K}_{\lambda^+}$ has λ^+-many realizations of p.*

Proof: If there are at most λ-many, then we can build $M \prec M_0 \precneqq M_1 \prec N$ with $\|M_0\| = \|M_1\|$ and $p(M_0) = p(M_1)$. By bigness, there is a nonalgebraic extension of p to M_0, so $M_0 \prec M_1$ is a (p, λ)-Vaughtian pair, a contradiction. †

This result suggests looking at resolutions. A crucial result of Grossberg and VanDieren allows us to show that models are saturated solely from the way they interact with a type that has no (p, λ)-Vaughtian pairs. We state with more assumptions than [Bal09, Lemma 13.9], but this is fine for the purpose at hand.

Definition 5.61. *Given $M \prec N$ from $\mathcal{K}_\lambda$ and $p \in gS(M)$, we say N* has a (p, λ, α)-decomposition of saturated models over M *iff there is a resolution $\langle N_i \in \mathcal{K}^{sat}_\lambda \mid i \leq \alpha\rangle$ such that*

(1) *$N_0 = M$ and $N_\alpha = N$; and*
(2) *for each $i < \alpha$, $p(N_{i+1}) - p(N_i)$ is nonempty.*

Theorem 5.62. *Suppose that $\mathcal{K}$ is an AEC with amalgamation that is weakly (χ, λ)-tame and weakly χ-stable. Let minimal $p \in gS(N_0)$ with $N_0 \in \mathcal{K}_\lambda$ that has no (p, λ)-Vaughtian pairs and $\alpha = \lambda \cdot \alpha$. If N_α admits a (p, λ, α)-decompostion of saturated models over N_0, then N_α realizes all types over N_0. N_0 is Galois saturated no (p, λ)-Vaughtian pairs*

Although we don't require the AEC to admit μ saturated unions, that will be implicit in the existence of the continuous chain of saturated models.

Proof: Let $\langle N_i \mid i \leq \alpha\rangle$ be the (p, λ, α)-decompostion and fix $q \in \mathrm{gS}(N_0)$.

We do some coding using the cardinal arithmetic: write $[1, \alpha)$ as a disjoint unions of $\langle S_i \mid i < \alpha\rangle$ such that $|S_i| = \lambda$ and $\min S_i > i$. Now we are going to build increasing, continuous $\langle M_i, M'_i \in \mathcal{K}_\lambda \mid i \leq \alpha\rangle$ and f_i such that

(1) $M_0 = N_0$ and $f_0 = \mathrm{id}$;
(2) $f_i : N_i \to_{M_0} M'_i$ with $f_i(N_i) = M_i$;
(3) for each i, $\{b_j \mid j \in S_i\}$ is an enumeration of $p(M'_i)$ with repetitions allowed; and
(4) $b_i \in M_{i+1}$.

This is enough: We had constructed $M_\alpha \prec M'_\alpha$ with $p(M_\alpha) = p(M'_\alpha)$. Since N_0 is saturated, p has a nonalgebraic extension of M_α. Since there are no (p, λ)-Vaughtian pairs, $M_\alpha = M'_\alpha$. By construction, M'_α realizes q and $M_\alpha \cong_{M_0} N_\alpha$. Thus, N_α realizes q. Since q was arbitrary, we have that N_α realizes all types over N_0.

Construction: Note at each stage of the construction, we pick some enumeration of $p(M'_i)$.

<u>$i = 0$:</u> Pick $M'_0 \succ M_0$ to realize both p and q.

<u>i limit:</u> Take unions.

<u>$i = j + 1$:</u> If $b_i \in M_{i+1}$, then we are done. If not, we know that there is some $c \in N_{i+1} - N_i$ realizing p. Since everything is Galois saturated, there is a unique nonalgebraic extension of the minimal p to M_i, so

$$\mathrm{gtp}(b/M_i; M'_i) = f_i^{-1} (\mathrm{gtp}(c/N_i; N_{i+1}))$$

Then we can find $M'_{i+1} \succ M'_i$ and $f_{i+1} : N_{i+1} \to M'_{i+1}$ such that $f_{i+1} \upharpoonright N_i = f_i$ and $f_{i+1}(c) = b$. Then setting $M_{i+1} = f_{i+1}(N_{i+1})$ works. †

The following is our first taste of using the nonexistence of Vaughtian pairs to transfer categoricity.

Lemma 5.63. *Suppose that $\mathcal{K}$ is an AEC that is weakly (χ, μ)-tame, χ-Galois stable, and satisfies $\mathcal{K}_\mu = \mathcal{K}^{sat}_\mu$ for $\chi < \mu$. If there is $M \in \mathcal{K}_\mu$ with minimal $p \in gS(M)$ that has no (p, μ)-Vaughtian pairs, then $\mathcal{K}^{sat}_{\mu^+} = \mathcal{K}_{\mu^+}$.*

Proof: Let $N \in \mathcal{K}_{\mu^+}$. We want to show this is Galois saturated, so let $M_0 \prec N$ of size μ. Since every model in $\mathcal{K}_\mu$ is Galois saturated, there is $f : M \cong M_0$ and there are no $(f(p), \mu)$-Vaughtian pairs. Fix $\alpha = \mu \cdot \alpha$. By Proposition 5.60, $f(p)$ is realized μ^+-many times in N. Thus, we can build $N' \prec N$ with an $(f(p), \mu, \alpha)$-decomposition of saturated models over M (note that the Galois saturation comes from the hypothesis $\mathcal{K}_\mu = \mathcal{K}^{sat}_\mu$). By Theorem 5.62, N' realizes all types over M and, thus, so does N. Since M_0 was arbitrary, N is Galois saturated. Thus $\mathcal{K}^{sat}_{\mu^+} = \mathcal{K}_{\mu^+}$.†

Lemma 5.64. *Suppose that $\mathcal{K}$ is an AEC that is weakly (χ, μ)-tame, χ-Galois stable, and satisfies $\mathcal{K}_\mu = \mathcal{K}^{sat}_\mu$ for $\chi < \mu$. Let $M \in \mathcal{K}^{sat}$ with minimal $p \in gS(M)$ and $\|M\| < \mu$ that has no (p, δ)-Vaughtian pairs. If $N \in \mathcal{K}_\mu$, then there is a minimal $q \in gS(N)$ that has no (q, μ)-Vaughtian pairs.*

Proof: Let $M_0 \prec N$ be Galois saturated of size δ. Then there is $f : M \cong M_0$. Since N is Galois saturated, there is a minimal $q \in \mathrm{gS}(N)$ that extends $f(p)$. If there were a (q, μ)-Vaughtian pair, there would be a $(f(p), \delta)$-Vaughtian pair by Proposition 5.59, which contradicts the nonexistence of (p, δ)-Vaughtian pairs. †

The following is our main lemma for categoricity transfer.

Lemma 5.65. *Suppose $\mathcal{K}$ is an AEC with amalgamation that is weakly (χ, κ)-tame and λ-categorical with $\chi < \delta < \lambda < \kappa$. If there is minimal $p \in gS(M)$ with no (p, δ)-Vaughtian paris for $M \in \mathcal{K}^{sat}_\delta$, then $\mathcal{K}^{sat}_{[\lambda,\kappa^+]} = \mathcal{K}_{[\lambda,\kappa^+]}$.*

Proof: We work by induction on $\kappa' \in [\lambda, \kappa^+]$. Note that by Theorem 5.55, $\mathcal{K}$ admits δ saturated unions.

- $\underline{\kappa' = \lambda}$: By assumption.
- $\underline{\kappa' \text{ is limit}}$: Here we crucially use that we're proving categoricity through saturation. Let $N \in \mathcal{K}_{\kappa'}$ and $M \prec N$ with $\lambda \leq \|M\| < \kappa'$. Since κ' is limit, we can find $M \prec M' \prec N$ with $\|M'\| = \|M\|^+$. By induction, M' is Galois saturated and so realizes every type over M. Thus, N realizes every type over M.
- $\underline{\kappa' = \mu^+}$: By Lemma 5.64, there is $N \in \mathcal{K}^{sat}_\mu$ and minimal $p \in \mathrm{gS}(N)$ with no (p, μ)-Vaughtian pairs. Then by Lemma 5.63, $\mathcal{K}^{sat}_{\mu^+} = \mathcal{K}_{\mu^+}$.

†

The final piece is to be able to transfer Vaughtian pairs freely once we exceed the Hanf number of the domain in size. This is weaker than Baldwin's [Bal09, Theorem 14.12] (which seems to be a version of [She99, Theorem 9.5.$(*)_9$]), but it is enough.

Theorem 5.66. *Suppose $\mathcal{K}$ is an AEC with amalgamation that is categorical in λ. If $M \in \mathcal{K}_\theta$ with $p \in gS(M)$ such that there is a true $\left(p, \beth_{(2^\theta)^+}\right)$-Vaughtian pair and $\lambda > \beth_{(2^\theta)^+}$, then there is a (p, κ)-Vaughtian pair for all $\kappa \geq \theta$. Moreover, if $\kappa \leq \lambda$, there is a true (p, κ)-Vaughtian pair.*

This moreover is actually what we will need.

Proof: Let $M_0 \prec N_0$ be the true $\left(p, \beth_{(2^\theta)^+}\right)$-Vaughtian pair. By categoricity and Corollary 4.34, we can assume that $N_0 = EM_\tau\left(\beth_{(2^\theta)^+}, \Phi\right)$ and let $f : N_0 \cong M_0$ be an isomorphism for $\Phi \in \Upsilon_{\mathrm{LS}(\mathcal{K})}[\mathcal{K}]$. Now we expand $EM\left(\beth_{(2^\theta)^+}, \Phi\right)$ to a $\tau(\Phi) \cup \{P, Q, R, F, c_m\}_{m \in M}$-structure N^+ by

1. P is a unary relation with $P^{N^+} = M_0$;
2. Q is a unary relation with $Q^{N^+} = M$ and $c_m^{N^+} = m$; and
3. F is a unary function with $F^{N^+} = f$.

This language has size θ and N^+ has size $\beth_{(2^\theta)^+}$, so we can find a blueprint Ψ patterned on the generating sequence $\beth_{(2^\theta)^+}$. Let $\kappa \geq \theta$ and consider $N_1 := EM_\tau(\kappa, \Psi)$ and $M_1 := P^{EM_\tau(\kappa,\Psi)} \restriction \tau$. Then $M_1 \prec N_1$ because Φ came from the presentation theorem.

Since $M_0 \succ M$ is Galois saturated, it has $\|M_0\|$-many realizations of p. Since $\|M_0\|$ is the Hanf number for $\|M\|$, a standard blueprint argument shows that there are arbitrarily many realizations of p. This means that p has a nonalgebraic extension to any model using amalgamation. In particular, it has a nonalgebraic extension to M_1.

Finally, we wish to show that $p(M_1) = p(N_1)$. Suppose $a \in p(N_1) = EM_\tau(\kappa, \Psi)$. Then a is some term of ordinals in $\tau(\Psi)$; moreover, we can remove any use of F. Then it's Galois type over M is realized in N_0. But then, Ψ forces this term to satisfy P, and so $a \in M_1$.

For the moreover, if $\kappa \leq \lambda$ we can use Corollary 4.34 to conclude that N_1 is Galois saturated, and therefore so is M_1 (since they are isomorphic), so the Vaughtian pair is true. †

5.6. **Putting it all together.** Now we are ready to prove our main theorems,

Theorem 5.67 ([GV06a,])**.** *If $\mathcal{K}$ has amalgamation, no maximal models, is κ-tame, and is categorical in λ^+ above $LS(\mathcal{K})^+ + \kappa$, then $\mathcal{K}$ is categorical in every $\chi \geq \lambda^+$.*

Proof: By Theorem 5.55, $\mathcal{K}$ admits λ saturated unions. By λ-Galois stability (which follows from categoricity by Theorem **??**), there is $M \in \mathcal{K}^{sat}_\lambda$. By Theorem 5.34, there is minimal $p \in \mathrm{gS}(M)$. By Proposition 5.57, there is no true (p, λ)-Vaughtian pair. Now we can apply Lemma 5.65 to conclude $\mathcal{K}_{\geq\lambda^+} = \mathcal{K}^{sat}_{\geq\lambda^+}$. †

Recall from Definition 5.44 that $\chi(\mathcal{K})$ denotes the cardinal so $\mathcal{K}$ is weakly $\big(\chi(\mathcal{K}), [\beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}, \lambda)\big)$-tame.

Theorem 5.68 ([She99,])**.** *Let $\mathcal{K}$ be an AEC with amalgamation that is categorical in $\lambda^+ > \beth_{(2^{\chi(\mathcal{K})})^+}$. Then $\mathcal{K}$ is categorical in all $\mu \in \big[\beth_{(2^{\chi(\mathcal{K})})^+}, \lambda^+\big]$.*

Proof: For the first part, recall from the definition that $\mathcal{K}$ is weakly $\big(\chi(\mathcal{K}), [\beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}, \lambda^+)\big)$-tame. Since $\beth_{(2^{\chi(\mathcal{K})})^+}$ is a $\chi(\mathcal{K})$ collection cardinal, by Theorem 5.46, $\mathcal{K}$ is $\beth_{(2^{\chi(\mathcal{K})})^+}$-categorical and this model is Galois saturated.

Let $M \in \mathcal{K}_{\beth_{(2^{\chi(\mathcal{K})})^+}}$ by this model. Let $M_0 \prec M$ of size $\chi(\mathcal{K})$ be Galois saturated. By Theorem 5.34, there is a minimal $p_0 \in \mathrm{gS}^{na}(M_0)$. By Corollary 5.26, p_0 has a nonalgebraic extension $p \in \mathrm{gS}^{na}(M)$ that does not $\chi(\mathcal{K})$-Galois split over M_0. By Corollary 5.58, this is minimal. The following claim will finish.

Claim: There is no $\big(p, \beth_{(2^{\chi(\mathcal{K})})^+}\big)$-Vaughtian pair.

Proof: For sake of contradiction, assume that $M_1 \prec N_1$ is a $\big(p, \beth_{(2^{\chi(\mathcal{K})})^+}\big)$-Vaughtian pair. By Proposition 5.59, this is a $\big(p_0, \beth_{(2^{\chi(\mathcal{K})})^+}\big)$-Vaughtian pair as well. By the categoricity in $\beth_{(2^{\chi(\mathcal{K})})^+}$, this is a true Vaughtian pair. Now we can apply Theorem 5.66 to get a true (p_0, λ)-Vaughtian pair $M_1 \prec N_1$. Again, we can extend the minimal p_0 to $p_1 \in gS^{na}(M_1)$ that does not $\chi(\mathcal{K})$-Galois split over M_0, so p_1 is minimal.

We claim that $M_1 \prec N_1$ are also a true (p_1, λ)-Vaughtian pair. Since $p_0 \leq p_1$, if If not, then there is $b \in p_1(N_1) - p_1(M_1)$. But $p_0 \leq p_1$, so $p_1(N_1) \subset p_0(N_1) \subset M_1$.

However, by Theorem 5.55, $\mathcal{K}$ admits λ saturated unions. This means we can invoke Proposition 5.57 and categoricty to conclude that there are no true (p_1, λ)-Vaughtian pairs, a contradiction. $\dagger_{\text{Claim}}$

By Theorem 5.55, $\mathcal{K}$ admits $\chi(\mathcal{K})^+$ saturated unions. Now we apply Lemma 5.65 to conclude categoricity on the interval $\big[\beth_{(2^{\chi(\mathcal{K})})^+}, \lambda^+\big]$ †

Theorem 5.69 ([Bon14, Theorem 7.4])**.** *If there are class-many almost strongly compact cardinals, then Shelah's Eventual Categoricity Conjecture for Successors is true.*

Proof: Set μ_λ to be the successor of first almost strongly compact above λ. Suppose $\mathcal{K}$ is an AEC with $\text{LS}(\mathcal{K}) \leq \lambda$ that is categorical in $\kappa^+ > \mu_\lambda$. By Theorem 4.6, $\mathcal{K}$ is $< \mu_\lambda$-tame. By [?], $(\lambda^+)^{<\mu_\lambda} = \lambda^+$, so by Theorem 4.9.(1), $\mathcal{K}_{\geq \mu_\lambda}$ has amalgamation, joint emebedding and no maximal models. By combining Theorems 5.68 and 5.67, $\mathcal{K}$ is categorical in all $\chi \geq \beth_{\left(2^{\chi(\mathcal{K})}\right)^+}$. Since $\text{LS}(\mathcal{K}) < \mu_\lambda$, $\mathcal{K}$ is categorical everywhere above μ_λ. †

References

[Bal09] John Baldwin, *Categoricity*, University Lecture Series, no. 50, American Mathematical Society, 2009.

[BB17] Will Boney and John Baldwin, *Hanf numbers and presentation theorems in abstract elementary classes*, Beyond First-Order Model Theory (Jose Iovino, ed.), CRC Press, 2017, pp. 327–352.

[BBHU08] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis, vol. 2 (Zoe Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, eds.), Cambridge University Press, 2008.

[BG17] Will Boney and Rami Grossberg, *Forking in short and tame abstract elementary classes*, Annals of Pure and Applied Logic **168** (2017), no. 8, 1517–1551.

[BKV06] John Baldwin, David Kueker, and Monica VanDieren, *Stability transfer for tame abstract elementary classes*, Notre Dame Journal of Formal Logic **6** (2006), 25–49.

[BL71] John Baldwin and Alistair Lachlan, *On strongly minimal sets*, Journal of Symbolic Logic **36** (1971), 79–96.

[Bona] Will Boney, *The Γ-ultraproduct and averageable classes*, Submitted, `http://arxiv.org/abs/1511.00982`.

[Bonb] ______, *Math 255 - classification theory in tame abstract elementary classes*, `http://math.harvard.edu/~wboney/fall17/`.

[Bonc] ______, *Shelah's omitting types theorem*, Notes, `http://math.harvard.edu/~wboney/notes/SOTTNotes.pdf`.

[Bond] ______, *Some model theory of algebraically closed valued fields with fixed value group*, In preparation.

[Bone] ______, *Tameness and abstract elementary classes*, http://math.harvard.edu/ wboney/BoneyTameSurvey.pdf.

[Bonf] ______, *Zilber's pseudo-exponential fields*, Notes, `http://math.harvard.edu/~wboney/notes/ZPEFNotes.pdf`.

[Bon14] ______, *Tameness from large cardinal axioms*, Journal of Symbolic Logic **79** (2014), no. 4, 1092–1119.

[Bon17] ______, *Computing the number of types of infinite length*, Notre Dame Journal of Formal Logic **58** (2017), no. 1, 133–154.

[BS08] John Baldwin and Saharon Shelah, *Examples of non-locality*, Journal of Symbolic Logic **73** (2008), 765–782.

[BU17] Will Boney and Spencer Unger, *Large cardinal axioms from tameness in abstract elementary classes*, Proceedings of the American Mathematical Society **145** (2017), no. 10, 4517–4532.

[BV17a] Will Boney and Sebastien Vasey, *Chains of saturated models in abstract elementary classes*, Archive for Mathematical Logic **56** (2017), no. 3, 187–213.

[BV17b] ______, *A survey of tame abstract elementary classes*, Beyond First-Order Model Theory (Jose Iovino, ed.), CRC Press, 2017, pp. 353–.

[CK12] C. C. Chang and H. Jerome Keisler, *Model theory*, 3rd ed., Dover Publications, 2012.

[ER56] Paul Erdős and Richard Rado, *A partition calculus in set theory*, Bulletin of the American Mathematical Society **62** (1956), 427–489.

[GHS] Vincent Guingona, Cameron Hill, and Lynn Scow, *Characterizing model-theoretic dividing lines via collapse of generalized indiscernibles*, Annals of Pure and Applied Logic, to appear.

[Grä68] George Grätzer, *Universal algebra*, Van Nostrand, 1968.

[Gro02] Rami Grossberg, *Classification theory for abstract elementary classes*, Logic and Algebra (Yi Zhang, ed.), vol. 302, American Mathematical Society, 2002, pp. 165–204.

[Gro1X] ______, *A course in model theory*, In preparation, 201X.

[GV06a] Rami Grossberg and Monica VanDieren, *Categoricity from one successor cardinal in tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), 181–201.

[GV06b] ______, *Galois-stability for tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.

[GV06c] ______, *Shelah's categoricity conjecture from a success for tame abstract elementary classes*, Journal of Symbolic Logic **71** (2006), 553–568.

[Här62] Klaus Härtig, *Über einen Quantifikator mit zwei Wirkungsbereichen.*, Colloquium on the foundations of mathematics, mathematical machines and their applications (L. Kalmár, ed.), Akadémiai Kiadó, 1962, pp. 31–36.

[Hod93] Wilfrid Hodges, *Model theory*, Encyclopedia of Mathematics and Its Applications, vol. 42, Cambridge University Press, 1993.

[Kei71] H. Jerome Keisler, *Model theory for infinitary logic*, Studies in Logic and the Foundations of Mathematics, vol. 62, North-Holland Publishing Company, 1971.

[KS96] Oren Kolman and Saharon Shelah, *Categoricity of theories in $\mathbb{L}_{\kappa,\omega}$, when κ is measurable cardinal, part 1*, Fundamenta Mathematica **151** (1996), 209–240.

[MM77] Menachem Magidor and Jerome Malitz, *Compact extensions of $L(Q)$, part 1a*, Annals of Mathematical Logic **11** (1977), 217–261.

[Mor65a] Michael Morley, *Categoricity in power*, Transactions of the American Mathematical Society **114** (1965), 514–538.

[Mor65b] ______, *Omitting classes of elements*, The theory of models, Proceedings of the 1963 International Symposium at Berkeley (John Addison, Leon Henkin, and Alfred Tarski, eds.), Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, 1965, pp. 265–273.

[MS76] Angus Macintyre and Saharon Shelah, *Uncountable universal locally finite groups*, Journal of Algebra **43** (1976), no. 1, 168–175.

[MS90] Michael Makkai and Saharon Shelah, *Categoricity of theories in $\mathbb{L}_{\kappa,\omega}$, with κ a compact cardinal*, Annals of Pure and Applied Logic **47** (1990), 41–97.

[MV11] Menachem Magidor and Jouko Väänänen, *On the Löwenheim-Skolem-Tarski numbers for extensions of first-order logic*, Journal of Mathematical Logic **11** (2011), 87–113.

[She] Saharon Shelah, *Abstract elementary classes near $\aleph_1$*, Chapter I of [She09b].

[She74] ______, *Categoricity of uncountable theories*, Proceedings of the Tarski Symposium, An international symposium held to honor Alfred Tarski on the occasion of his seventieth birthday (Leon Henkin, ed.), Proceedings of symposia in pure mathematics, vol. 25, American Mathematical Society, 1974, pp. 187–203.

[She75] ______, *Generalized quantifiers and compact logic*, Transactions of the American Mathematical Society **204** (1975), 342–364.

[She87a] ______, *Classification of nonelementary classes, II. Abstract elementary classes*, Classification theory (John Baldwin, ed.), 1987, pp. 419–197.

[She87b] ______, *Universal classes*, Classification theory (John Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 264–418.

[She99] ______, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), 261–294.

[She01] ______, *Categoricity of an abstract elementary class in two successive cardinals*, Israel Journal of Mathematics **126** (2001), 29–128.

[She09a] ______, *Categoricity in abstract elementary classes: going up inductively*, Classification theory for abstract elementary classes, vol. 18, College Publications, 2009, pp. 224–377.

[She09b] ______, *Classification theory for abstract elementary classes*, Mathematical Logic and Foundations, vol. 18 & 20, College Publications, 2009.

[She17] ______, *Existentially closed locally finite groups*, Beyond First-Order Model Theory (Jose Iovino, ed.), CRC Press, 2017.

[SV99] Saharon Shelah and Andrés Villaveces, *Toward categoricity for classes with no maximal models*, Annals of Pure and Applied Logic **97** (1999), 1–25.

[Tar54] Alfred Tarski, *Contributions to the theory of models I*, Indagationes Mathematicae **16** (1954), 572–581.

[Vää79] Jouko Väänänen, *Abstract logic and set theory. I. Definability.*, Logic Colloquium 78 (Mons, 1978), Studies in Logic and the Foundations of Mathematics, vol. 97, North-Holland Publishing Company, 1979, pp. 391–421.

[Van06] Monica VanDieren, *Categoricity in abstract elementary classes with no maximal models*, Annals of Pure and Applied Logic **141** (2006), 108–147.

[Van16] ______, *Symmetry and the union of saturated models in superstable abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), 395–407.

[Vasa] Sebastien Vasey, *Tameness from two successive good frames*, Submitted, `https://arxiv.org/abs/1707.09008`.

[Vasb] ______, *Toward a stability theory of tame abstract elementary classes*, Submitted, `https://arxiv.org/abs/1609.03252`.

[Vas16a] ______, *Building independence relations in abstract elementary classes*, Annals of Pure and Applied Logic **167** (2016), no. 11, 1029–1092.

[Vas16b] ———, *Forking and superstability in tame aecs*, Journal of Symbolic Logic **81** (2016), 357–383.

[Vas16c] ———, *Infinitary stability theory*, Archive for Mathematical Logic **55** (2016), no. 3–4, 562–592.

[Vas17a] ———, *Saturation and solvability in abstract elementary classes with amalgamation*, Archive for Mathematical Logic **56** (2017), no. 5-6, 671–690.

[Vas17b] ———, *Shelah's eventual categoricity conjecture in universal classes: part I*, Annals of Pure and Applied Logic **168** (2017), no. 9, 1609–1642.

[Vas17c] ———, *Shelah's eventual categoricity conjecture in universal classes: part II*, Selecta Mathematica **23** (2017), no. 2, 1469–1506.

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